Chương 4

Trường tĩnh điện trong vật chất

Bài tập 4.1

$E = V/x = 500/10^{-3} = 5\times10^{5}$. Table 4.1: $\alpha/4\pi\varepsilon_0 = 0.66\times10^{-30}$, so

$\alpha = 4\pi(8.85\times10^{-12})(0.66\times10^{-30}) = 7.34\times10^{-41}.$

$p = \alpha E = ed \Rightarrow d = \alpha E/e = (7.34\times10^{-41})(5\times10^{5})/(1.6\times10^{-19}) =$ 2.29×10^{-16} m.

$d/R = (2.29\times10^{-16})/(0.5\times10^{-10}) = \boxed{4.6\times10^{-6}.}$ Để in hóa, chẳng hạn d = R. Thì R= $\alpha E/e = \alpha V/ex \Rightarrow V = R\ ex/\alpha$ $= (0.5\text{x}10^{-10})(1.6\text{x}10^{-19})(10^{-3})/(7.34\text{x}10^{-41}) =$ $\boxed{10^{8}V.}$

Bài tập 4.2

Đầu tiên tìm trường, ở bán kính r, dùng định luật Gauss: $\int E.da = \frac{1}{\varepsilon_0}Q_{enc}$, or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}Q_{enc}.$$

$$Q_{enc} = \int_0^r \rho dr = \frac{4\pi q}{\pi a^3}\int_0^r e^{-2\bar{r}/a}\bar{r}^2 d\bar{r} = \frac{4q}{a^3}\left[-\frac{a}{2}e^{-2\bar{r}/a}\left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2}\right)\right]\Big|_0^r$$

$$= -\frac{2q}{a^2}\left[e^{-2r/a}(r^2 + ar + \frac{a^2}{2})\right] = q\left[1 - e^{-2\bar{r}/a}(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2})\right].$$

[Chú ý: $Q_{enc}(r\to\infty) = q.$] Vì vậy trường của electron có thể là

$E_e = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left[1 - e^{-2r/a}(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2})\right]$. Proton sẽ bị dịch chuyển từ r = 0 đến điểm d ở đó E_e = E (trường ngoài):

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right].$$

Khai triển theo (d/a):

$$e^{-2d/a} = 1-\left(\frac{2d}{a}\right)+\frac{1}{2}\left(\frac{2d}{a}\right)^2-\frac{1}{3!}\left(\frac{2d}{a}\right)^3+\ldots = 1-2\frac{d}{a}+2\left(\frac{d}{a}\right)^2-\frac{4}{3}\left(\frac{d}{a}\right)^3+\ldots 1-e^{-2d/a}\left(1+2\frac{d}{a}+2\frac{d^2}{a^2}\right)$$

$$= 1-\left(1-2\frac{d}{a}+2\left(\frac{d}{a}\right)^2-\frac{4}{3}\left(\frac{d}{a}\right)^3+\ldots\right)\left(1+2\frac{d}{a}+2\frac{d^2}{a^2}\right)$$

$$= \gamma-\gamma-2\frac{d}{a}-2\frac{d^2}{a^2}+2\frac{d}{a}+4\frac{d^2}{a^2}+4\frac{d^3}{a^3}-2\frac{d^2}{a^2}-4\frac{d^3}{a^3}+\frac{4}{3}\frac{d^3}{a^3}+\ldots$$

$= \frac{4}{3}\left(\frac{d}{a}\right)^3 +$ các số hạng bậc cao.

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right) = \frac{1}{4\pi\varepsilon_0}\frac{4}{3a^3}(qd) = \frac{1}{3\pi\varepsilon_0 a^3}\,p.\quad \boxed{\alpha = 3\pi\varepsilon_0 a^3.}$$

[Không khác nhiều với mô hình hình cầu đồng đều của ví dụ.(Xem pt. 4.2). Chú ý rằng kết quả này tiên đoán $\frac{1}{4\pi\varepsilon_0}\alpha = \frac{3}{4}a^3 = \frac{3}{4}(0.5\times10^{-10})^3 = 0.09\times10^{30}m^3$, so với giá trị thực nghiệm (bảng 4.1) 0.66 x 10^{-30} m^3. Thật trớ trêu, các công thức cổ điển (phương trình. 4.2) khá gần với giá trị thực nghiệm.]

Bài tập 4.3

$\rho\ \ r\ = \mathrm{Ar}$. Điện trường (theo định luật Gauss):

$\oint E.da = E(4\pi^2) = \frac{1}{\varepsilon_0}Q_{enc} = \frac{1}{\varepsilon_0}\int_0^r A\vec{r}4\pi\vec{r}^2 d\vec{r}$, or E= $\frac{1}{4\pi r^2}\frac{4\pi A}{\varepsilon_0}\frac{r^4}{4} = \frac{Ar^2}{4\varepsilon_0}$. Trường “bên trong” này cân bằng với trường bên ngoài E khi hạt nhân “lệch tâm” một lượng d : $ad^2/4\varepsilon_0 = E \Rightarrow d = \sqrt{4\varepsilon_0 E/A}$. Vì vậy moment lưỡng cực cảm ứng là $p = ed = 2e\sqrt{\varepsilon_0/A}\sqrt{E}$. Hiển nhiên p tỉ lệ với $E^{1/2}$.

Đối với pt. 4.1 để đúng trong giới hạn trường yếu , E phải tỉ lệ với r , đối với r nhỏ , có nghĩa là nó phải tiến tới không đổi (khác 0) tại gốc tọa độ : $\boxed{\rho(0)\neq 0}$ (chứ không phải không xác định)

Bài tập 4.4

Trường của q : $\frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}$. Moment lưỡng cực cảm ứng của nguyên tử : $p = \alpha E = \frac{\alpha q}{4\pi\varepsilon_0 r^2} \hat{r}$.

Trường của lưỡng cực này , tại vị trí q ($\theta = \pi$, trong phương trình.3.103) :

$E = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \left(\frac{2\alpha q}{4\pi\varepsilon_0 r^2} \right)$ (phải) .

Lực tác dụng lên q do trường này: (hút) .

Bài tập 4.5

Trường của p_1 tại p_2 ($\theta = \pi/2$ trong phương trình. 3.103) : $E_1 = \frac{p_1}{4\pi\varepsilon_0 r^3} \hat{\theta}$ (hướng xuống).

Momen trên p_2 : $N_2 = p_2 \times E_1 = p_2 E_1 \sin 90^o = p_2 E_1 = \boxed{\frac{p_1 p_2}{4\pi\varepsilon_0 r^3}}$ (hướng vào trong trang giấy).

Momen trên p_1 : $N_1 = p_1 \times E_2 = \boxed{\frac{2 p_1 p_2}{4\pi\varepsilon_0 r^3}}$ (hướng vào trong trang giấy)

Bài tập 4.6

Dùng lưỡng cực ảnh được biểu diễn trong hình .(a). Kéo ngược lại, thay p_i ở gốc tọa độ, hình. (b).

$E_i = \frac{p}{4\pi\varepsilon_0 \; 2z^{\,3}} \; 2\cos\theta\hat{r} + \sin\theta\hat{\theta}$; $p = p\cos\theta\hat{r} + p\sin\theta\hat{\theta}$.

(ngoài trang).

Nhưng $\sin\theta\cos\theta = 1/2 \; \sin 2\theta$, so $\boxed{N = \frac{p^2 \sin 2\theta}{4\pi\varepsilon_0 \; 16z^3}}$ (ngoài trang)

Đối với $0 < \theta < \pi/2, N$ có khuynh hướng quay p ngược chiều kim đồng hồ ; đối với $\pi/2 < \theta < \pi, N$ quay p cùng chiều kim đồng hồ. Vì thế

$$\boxed{\text{Hướng ổn định vuông góc với bề mặt –hoặc } \uparrow or \downarrow.}$$

Bài tập 4.7

Giả sử rằng trường là đều và hướng theo trục y. Trươc hết mảnh p từ vô cùngdọc theo trục x - cái này không sinh công, vì F $\perp dl$. (Nếu E không đều, mảnh p nằm dọc theo quỹ đạo $\perp$ trường.) Bây giờ quay (ngược chiều kim đồng hồ) đến điểm cuối. Momen bị tác dụng bởi E là $N = p \times E = pE\sin\theta\hat{z}.$ Momen mà chúng ta tác dụng là $N = pE\sin\theta$ *theo chiều kim đồng hồ*, và $d\theta$ ngược chiều kim đồng hồ, vì vậy công toàn phần được thực hiện bởi chúng ta âm :

$$U = \int_{\pi/2}^{\theta} pE\sin\bar{\theta}\, d\bar{\theta} = pE\left(-\cos\bar{\theta}\right)\Big|_{\pi/2}^{\theta} = -pE\left(\cos\theta - \cos\frac{\pi}{2}\right) = -pE\cos\theta = -p.E. \quad qed$$

Bài tập 4.8

$$U = -p_1.E_2, but\ \ E_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(p_2.\hat{r})\hat{r} - p_2\right].\ \text{So}\ U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[p_1 p_2 - 3\left(p_1\hat{r}\right)\left(p_2.\hat{r}\right)\right]. \quad qed$$

Bài tập 4.9

$$(a)\ F = (p.\nabla)E\ (Eq.4.5);\ E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$F_z = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) \frac{q}{4\pi\varepsilon_0} \frac{x}{x^2 + y^2 + z^2 \;\; ^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0} \left\{ p \left[\frac{1}{x^2+y^2+z^2 \;\; ^{3/2}} - \frac{3}{2} x \frac{2x}{x^2+y^2+z^2 \;\; ^{5/2}} \right] + p_y \left[-\frac{3}{2} x \frac{2y}{x^2+y^2+z^2 \;\; ^{5/2}} \right] + p_z \left[\right. \right.$$

$$= \frac{q}{4\pi\varepsilon_0} \left[\frac{p_z}{r^3} - \frac{3x}{r^5} \;\; p_x x + p_y y + p_z z \;\; \right] = \frac{q}{4\pi\varepsilon_0} \left[\frac{p}{r^3} - \frac{3r \;\; p.r}{r^5} \right]_x .$$

$$F = \boxed{\frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \left[p - 3 \;\; p.\hat{r} \;\; \hat{r} \right].}$$

$b \quad E = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \;\; 3\left[p. \; -\hat{r} \;\; \right] -\hat{r} \; - p \;\; = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \left[3 \;\; p.\hat{r} \;\; \hat{r} - p \right].$ (Điều này xuất phát từ phương trình. 3.104; dấu trừ là bởi vì r hướng về phía p, trong bài tập này.)

$$F = qE = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \left[3 \;\; p.\hat{r} \;\; - p \right].$$

[Chú ý rằng lực bằng và ngược dấu, như bạn mong đợi tự định luật III Newton.]

Bài tập 4.10

(a) $\sigma_b = P.\hat{n} = \boxed{kR;} \;\; \rho_b = -\nabla.P = -\frac{1}{r^3} \frac{\partial}{\partial r} \;\; r^2 kr \;\; = -\frac{1}{r^2} 3kr^2 = \boxed{-3k.}$

(b) Đối với r < R , E= $\frac{1}{3\varepsilon_0} \rho r\hat{r}$ (Bài tập. 2.12) , vì vậy $E = \boxed{- \;\; k/\varepsilon_0 \;\; r.}$

Đối với r > R, tương tự nếu tất cả điện tích ở tâm ; nhưng
$Q_{tot} = kR \quad 4\pi R^2 \; + \; -3k \left(\frac{4}{3}\pi R^3\right) = 0$, vì vậy $\boxed{E = 0.}$

Bài tập 4.11

$\rho_b = 0; \sigma = P.\hat{n} = \pm P$ (dấu cộng ở một đầu sang đầu mà các điểm P hướng tới ; dấu trừ ở đầu còn lại sang đầu mà các điểm P hướng ra xa).

(i) $L >> a$. Do đó các đầu trong giống các điện tích điểm, và toàn bộ vật giống như lưỡng cực vật lý, chiều dài L và điện tích $P\pi a^2$. Xem hình. (a).

(ii) $L << a$. Thế thì nó giống như một tụ điện bản song song tròn. Trường gần đều bên trong ; "trường viền" không đều ở các biên. Xem hình. (b).

(iii) $L \approx a$. Xem hình .(c).

Bài tập 4.12

$V = \frac{1}{4\pi\varepsilon_0}\int \frac{P.\hat{r}}{r^2} dr = P.\left\{\frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{r^2} dr\right\}$. Nhưng các số hạng trong ngoặc nhọn đúng là trường của quả cầu tích điện đều, chia cho p. Tích phân được tính trong bài tập.2.7 và 2.8 :

$$\frac{1}{4\pi\varepsilon_0}\int\frac{\hat{r}}{r^2}dr=\frac{1}{p}\begin{cases}\dfrac{1}{4\pi\varepsilon_0}\dfrac{4/3\ \pi R^3 p}{r^2}\hat{r}, & r>R\ ,\\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{4/3\ \pi R^3 p}{R^3}\hat{r}, & r<R\ ,\end{cases}\quad \text{So } V\ r,\theta\ =\begin{cases}\dfrac{R^3}{3\varepsilon_0 r^2}P.\hat{r}=\boxed{\dfrac{R^3 P\cos\theta}{3\varepsilon_0 r^2}},\\ \dfrac{1}{3\varepsilon_0}P.r=\boxed{\dfrac{\Pr\cos\theta}{3\varepsilon_0}},\end{cases}$$

$$\left.\begin{matrix} r>R \\ \\ r<R\end{matrix}\right\}$$

Bài tập 4.13

Xem nó như hai hình trụ có mật độ điện tích đều ngược nhau $\pm\rho$. Bên trong trường, ở khoảng cách s từ trục của hình trụ tích điện đều được cho bởi định luật Gauss : $E2\pi sl=\frac{1}{\varepsilon_0}\rho\pi s^2 l\Rightarrow E=\ \rho/2\varepsilon_0\ s$. Đối với hai hình trụ như thế, một cộng và một trừ. Trường toàn phần (bên trong) là $E=E_+ + E_- =\ \rho/2\varepsilon_0\quad s_+ - s_-$. Nhưng $s_+ - s_- = -d$, vì vậy $E=\boxed{\rho d/\ 2\varepsilon_0\ ,}$ ở đây d là vector từ trục âm đến trục dương. Trong trường hợp này, momen lưỡng cực toàn phần của khoanh có chiều dài l bằng P $\pi a^2 l\ =\ \rho\pi a^2 l\ d$. Vì vậy $\rho d=P$, và $\boxed{E=-P/\ 2\varepsilon_0\ ,}$ đối với $s<a$.

Bên ngoài , định luật Gauss cho chúng ta $E2\pi sl=\frac{1}{\varepsilon_0}\rho\pi a^2 l\Rightarrow E=\frac{\rho a^2}{2\varepsilon_0}\frac{\hat{s}}{s}$, đối với một hình trụ. Để kết hợp, $E=E_+ + E_- =\frac{\rho a^2}{2\varepsilon_0}\left(\frac{\hat{s}_+}{s_+}-\frac{\hat{s}_-}{s_-}\right)$, ở đây

$$s_\pm = s\mp\frac{d}{2};$$

$$\frac{s_\pm}{s_\pm^2}=\left(s\mp\frac{d}{2}\right)\left(s^2+\frac{d^2}{4}\mp s.d\right)^{-1}\cong\frac{1}{s^2}\left(s\mp\frac{d}{2}\right)\left(1\mp\frac{s.d}{s^2}\right)^{-1}\cong\frac{1}{s^2}\left(s\mp\frac{d}{2}\right)\left(1\pm\frac{s.d}{s^2}\right)$$

$$=\frac{1}{s^2}\left(s\pm s\frac{s.d}{s^2}\mp\frac{d}{2}\right)$$

(chỉ giữ các số hạng bậc nhất của d)

$$\left(\frac{\hat{s}_+}{s_+}-\frac{\hat{s}_-}{s_-}\right)=\frac{1}{s^2}\left[\left(s+s\frac{s.d}{s^2}-\frac{d}{2}\right)-\left(s-s\frac{s.d}{s^2}+\frac{s}{2}\right)\right]=\frac{1}{s^2}\left(2\frac{s\ \ s.d}{s^2}-d\right).$$

$$\boxed{E\ \ s\ \ =\frac{a^2}{2\varepsilon_0}\frac{1}{s^2}\left[2\ \ P.\hat{s}\ \ \hat{s}-P\right],}\quad \text{đối với } s>a$$

Bài tập 4.14

Điện tích tổng cộng trong điện môi bằng $Q_{tot}=\oint_S \sigma_b da+\int_V \rho_b d\tau=\oint_S P.da-\int_V \nabla.P d\tau.$

Nhưng định lí divergence nói rằng $\oint_S P.da=\int_V \nabla.P d\tau,$ so $Q_{enc}=0.$ qed

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Bài tập 4.15

(a) $\rho_b=-\nabla.P=-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right)=-\frac{k}{r^2};$

$$\sigma_b=P.\hat{n}=\begin{cases}+P.\hat{r}=k/b & r=b,\\ -P.\hat{r}=-k/a & r=a.\end{cases}$$

Định luật Gauss $\Rightarrow E=\frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}\hat{r}.$ Đối với r < a , Q_{enc} = 0 , so $\boxed{E=0}$. Đối với r > b , Q_{enc} =0 (Bài tập. 4.14) , vì vậy $\boxed{E=0}$

Đối với a < r< b , $Q_{enc}=\left(\frac{-k}{a}\right)\ 4\pi a^2\ +\int_a^r\left(\frac{-k}{\bar{r}^2}\right)4\pi\bar{r}^2 d\bar{r}=-4\pi ka-4\pi k\ \ r-a\ \ =-4\pi kr;$ vì vậy $\boxed{E=-\ \ k/\varepsilon_0 r\ \ \hat{r}.}$

(b) $\oint D.da=Q_{f_{enc}}=0\Rightarrow D=0$ ở mọi nơi. $D=\varepsilon_0 E+P=0\Rightarrow E=\ \ -1/\varepsilon_0\ \ P,$ vì vậy

$\boxed{E=0 \text{ (đối với r < a và r > b);}}$ $\boxed{E=\ \ k/\varepsilon_0 r\ \ \hat{r} \text{ (đối với a < r < b)}}$

Bài tập 4.16

(a) Giống như E_0 trừ trường ở tâm hình cầu với độ phân cực đều P. Cái sau bằng (pt. 4.14) $-P/3\varepsilon_0$. Vì vậy $\boxed{E = E_0 + \frac{1}{3\varepsilon_0}P}$

$D = \varepsilon_0 E = \varepsilon_0 E_0 + \frac{1}{3}P = D_0 - p + \frac{1}{3}p$, So $\boxed{D = D_0 - \frac{2}{3}P.}$

(b) Giống như E_0 trừ trường của $\pm$ các điện tích tại hai đầu của "kim" – nhưng những cái này nhỏ , và cách xa, vì vậy $\boxed{E=E_0}$

$$\boxed{D = \varepsilon_0 E = \varepsilon_0 E_0 = D_0 - P, so \boxed{D = D_0 - P.}}$$

(c) Giống như E_0 trừ trường của tụ điện bản song song với bảng cao hơn tại $\sigma = P$. Cái sau bằng $\boxed{E = E_0 + \frac{1}{\varepsilon_0}P}$ $-\ 1/\varepsilon_0\ \ P,$ s $D = \varepsilon_0 E = \varepsilon_0 E_0 + P, so \boxed{D = D_{0.}}$ o

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Bài tập 4.18

(a) Áp dụng $\int D.da = Q_{f_{enc}}$ cho bề mặt Gauss được biểu diễn. $DA = \sigma A \Rightarrow \boxed{D = \sigma.}$
(Chú ý : D =0 bên trong mảnh kim loại.) Điều này đúng ở cả hai tấm; D hướng xuống.

(b) $D = \varepsilon E \Rightarrow E = \sigma/\varepsilon_1$ ở tấm 1 , $E = \sigma/\varepsilon_0$ ở tấm 2 . Nhưng $\varepsilon = \varepsilon_0\varepsilon_r$, vì vậy

$\varepsilon_1 = 2\varepsilon_0; \varepsilon_2 = \frac{3}{2}\varepsilon_0.\boxed{E_1 = \sigma/2\varepsilon_0,}\boxed{E_2 = 2\sigma/3\varepsilon_0.}$

(c) $p = \varepsilon_0\chi_e E,$ vì vậy $P = \varepsilon_0\chi_e d\big/\ \varepsilon_0\varepsilon_r\ \ =\ \ \chi_e/\varepsilon_r\ \ ;\chi_e = \varepsilon_r - 1 \Rightarrow P =\ 1 - \varepsilon_r^{-1}\ \ \sigma\ \ .$

$\boxed{P_1 = \sigma/2,}\boxed{P_2 = \sigma/3.}$

(d) $V = E_1 a + E_2 a =\ \ \sigma a/6\varepsilon_0\ \ \ 3+4\ \ = \boxed{7\sigma a/6\varepsilon_0.}$

(e) $\rho_b = 0;\ \ \sigma_b = +P_1$ ở bên dưới tấm $\ 1\ = \sigma/2,$ $\quad \sigma_b = +P_2$ ở bên dưới tấm $\ 2\ = \sigma/3,$

$\sigma_b = -P_1$ ở trên tấm $\ 1\ = -\sigma/2,$ $\quad \sigma_b = +P_2$ ở trên tấm $\ 2\ = -\sigma/3,$

(f) Ở tấm1 : Điện tích bề mặt toàn phần bên trên : $\sigma -\ \ \sigma/2\ \ = \sigma/2$

Điện tích bề mặt toàn phần bên dưới:

$\sigma/2 \;-\; \sigma/3 \;+\; \sigma/3 \;-\sigma = -\sigma/2,$

Ở tấm 2 : điên tích bề mặt toàn phần bên trên : $\sigma - \;\sigma/2 \;+\; \sigma/2 \;-\; \sigma/3 \; = 2\sigma/3,$

Điện tích bề mặt toàn phần bên dưới : $\sigma/3 \;-\sigma = -2\sigma/3,$

Bài tập 4.19

Khi không có điện môi, $C_0 = A\varepsilon_0/d$ (Pt. 2.54).

Trong cấu hình (a) , với $+\sigma$ ở bảng trên , $-\sigma$ bên dưới , $D = \sigma$ giữa các bảng .

$E = \sigma/\varepsilon_0$ (trong không khí) và $E = \sigma/\varepsilon$ (trong điện môi). Vì vậy

$$V = \frac{\sigma}{\varepsilon_0}\frac{d}{2} + \frac{\sigma}{\varepsilon}\frac{d}{2} = \frac{Qd}{2\varepsilon_0 A}\left(1 + \frac{\varepsilon_0}{\varepsilon}\right).$$

$$C_a = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}\left(\frac{2}{1 + 1/\varepsilon_r}\right) = \boxed{\frac{C_a}{C_0} = \frac{2\varepsilon_0}{1+\varepsilon_r}.}$$

Trong cấu hình (b) , với sự chênh lệch thế V : $E = V/d$, so $\sigma = \varepsilon_0 E = \varepsilon_0 V/d$ (trong không khí).

$P = \varepsilon_0 \chi_e V/d$ (trong điện môi) , vì vậy $\sigma_b = -\varepsilon_0 \chi_e V/d$ (tại mặt trên của điện môi).

$\sigma_{tot} = \varepsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \varepsilon_0 \chi_e V/d$, so $\sigma_f = \varepsilon_0 V \; 1 + \chi_e \; /d = \varepsilon_0 \varepsilon_r V/d$ (ở mảnh trên điện môi).

$$\Rightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\varepsilon_0\frac{V}{d} + \varepsilon_0\frac{V}{d}\varepsilon_r\right) = \frac{A\varepsilon_0}{d}\left(\frac{1+\varepsilon_r}{2}\right). \boxed{\frac{C_b}{C_0} = \frac{1+\varepsilon_r}{2}.}$$

[Cái nào lớn hơn ?

$$\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\varepsilon_r}{2} - \frac{2\varepsilon_r}{1+\varepsilon_r} = \frac{1+\varepsilon_r{}^2 - 4\varepsilon_r}{2\;1+\varepsilon_r} = \frac{1+2\varepsilon_r + 4\varepsilon_r^2 - 4\varepsilon_r}{2\;1+\varepsilon_r} = \frac{1-\varepsilon_r{}^2}{2\;1+\varepsilon_r} > 0.$$ So $C_b > C_a$]

Nếu trục x hướng xuống:

Bài tập 4.20

$\int D.da = Q_{f_{enc}} \Rightarrow D4\pi r^2 = \rho \frac{4}{3}\pi r^3 \Rightarrow D = \frac{1}{3}\rho r \Rightarrow E = \rho r/3\varepsilon\ \hat{r}$, đối với

$r < R; D4\pi r^2 = \rho \frac{4}{3}\pi R^3 \Rightarrow D = \rho R^3/3r^2 \Rightarrow E = \rho R^3/3\varepsilon_0 r^2\ \hat{r},$ for $r > R$.

$$V = -\int_\infty^0 E.dl = \frac{\rho R^3}{3\varepsilon_0}\frac{1}{r}\Big|_\infty^R - \frac{\rho}{3\varepsilon}\int_R^0 r dr = \frac{\rho R^2}{3\varepsilon_0} + \frac{\rho}{3\varepsilon}\frac{R^2}{2} = \boxed{\frac{\rho R^2}{3\varepsilon_0}\left(1 + \frac{1}{2\varepsilon_r}\right).}$$

Bài tập 4.21

Đặt Q là điện tích chiều dài *l* của vật dẫn bên trong.

$$\oint D.da = D2\pi sl = Q \Rightarrow D = \frac{Q}{2\pi sl}; \quad E = \frac{Q}{2\pi\varepsilon_0 sl}\ \ a < s < b\ , \quad E = \frac{Q}{2\pi\varepsilon sl}\ \ (b < r < c).$$

$$V = -\int_c^a E.dl = \int_a^b \left(\frac{Q}{2\pi\varepsilon_0 l}\right)\frac{ds}{s} + \int_b^c \left(\frac{Q}{2\pi\varepsilon l}\right)\frac{ds}{s} = \frac{Q}{2\pi\varepsilon_0 l}\left[\ln\left(\frac{b}{a}\right) + \frac{\varepsilon_0}{\varepsilon}\ln\left(\frac{c}{b}\right)\right].$$

$$\frac{C}{l} = \frac{Q}{Vl} = \boxed{\frac{2\pi\varepsilon_0}{\ln(b/a) + (1/\varepsilon_r)\ln(c/b)}.}$$

Bài tập 4.22

Phương pháp tương tự như ví dụ. 4.7 : Giải phương trình Laplace đối với $V_{in}(s,\phi)$ $(s < a)$ và $V_{out}(s,\phi)$ $(s > a)$, thõa điều kiện biên.

(i) $V_{in} = V_{out}$ tại $s = a$,

(ii) $\varepsilon \frac{\partial V_{out}}{\partial s}$ tại $s = a$,

(iii) $V_{out} \to -E_0 s\cos\phi$ đối với $s >> a$.

Từ bài tập. 3.23 (dẫn ra điều kiện biên (iii)):

$$V_{in}(s,\phi) = \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi) \quad .$$

(Tôi đã khử các số hạng hằng số bằng cách đặt V = 0 trên mặt phẳng *y z*.) Điều kiện (i) nói

$$\sum a^k (a_k \cos k\phi + b_k \sin k\phi) = -E_0 s\cos\phi + \sum a^{-k}(c_k \cos k\phi + d_k \sin k\phi) ,$$

Trong khi (ii) nói

$$\varepsilon_r \sum ka^{k-1} \ a_k \cos k\phi + b_k \sin k\phi \ = -E_0 \cos\phi - \sum ka^{-k-1} \ c_k \cos k\phi + d_k \sin k\phi \ .$$

Hiển nhiên $b_k = d_k = 0$ đối với tất cả k , $a_k = c_k = 0$ nếu k khác 1 , trong khi đó k = 1 ,

$$aa_1 = -E_0 a + a^{-1} c_1, \ \ \varepsilon_r a_1 = -E_0 - a^{-2} c_1 \ .$$

Giải tìm a_1 ,

$$a_1 = -\frac{E_0}{1+\chi_e/2}, \text{ vì vậy } V_{in} \ s,\phi \ = -\frac{E_0}{1+\chi_e/2} s\cos\phi = -\frac{E_0}{1+\chi_e/2} x,$$

Và vì thế $E_{in} \ s,\phi \ = -\frac{\partial V_{in}}{\partial x}\hat{x} = \boxed{\frac{E_0}{1+\chi_e/2}}$. Như trong trường hợp hình cầu (Ví dụ. 4.7) , trường bên trong là đều.

Bài tập 4.23

$$P_0 = \varepsilon_0 \chi_e E_0; E_1 = -\frac{1}{3\varepsilon_0} p_0 = -\frac{\chi_e}{3} E_0; P_1 = \varepsilon_0 \chi_e E_1 = -\frac{\varepsilon_0 \chi_e^2}{3} E_0; E_2 = -\frac{1}{3\varepsilon_0} P_1 = \frac{\chi_e}{9} E_0; \ \ldots$$

Hiển nhiên $E_n = \left(-\frac{\chi_e}{3}\right)^n E_0$, vì vậy

$$E = E_0 + E_1 + E_2 + \ldots = \left[\sum_{n=0}^{\infty} \left(-\frac{\chi_e}{3}\right)^n\right] E_0 \ .$$

Chuỗi hình học có thể được cộng một cách tường minh :

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ so } \boxed{E = \frac{1}{1+\chi_e/3} E_0},$$

Phù hợp với pt. 4.49. [Kì lạ thay , thường thì phương pháp này đòi hỏi rằng $\chi_e < 3$ (ngược lại các chuỗi không xác định phân kì) , tuy nhiên kết quả không chịu hạn chế như thế, bởi vì chúng ta cũng có thể nhận được nó bằng phương pháp của ví dụ. 4.7.]

Bài tập 4.24

Thế :

$$\begin{cases} V_{out}\left(r,\theta\right)=-E_0 r\cos\theta+\sum\frac{B_l}{r^{l+1}}P_l\left(\cos\theta\right), \\ V_{med}\left(r,\theta\right)=\sum\left(A_t r^t+\frac{B_l}{r^{l+1}}\right)P_l\left(\cos\theta\right), \quad (\text{ a < r <b }) \\ V_{in}\left(r,\theta\right)=0, \end{cases}$$

Các điều kiện biên :

$$\begin{cases} \left(i\right) V_{out}=V_{med},(r=b); \\ (ii) \qquad \frac{\partial V_{med}}{\partial r}=\varepsilon_0\frac{\partial V_{out}}{\partial r} \text{ , (r = b) ;} \\ (iii) V_{med}=0(r=a) \end{cases}$$

$$\text{(i)}\Rightarrow -E_0 b\cos\theta+\sum\frac{B_l}{b^{l+1}}P_l\left(\cos\theta\right)=\sum\left(A_l b^l+\frac{\bar{B}_l}{b^{l+1}}\right)P_l\left(\cos\theta\right);$$

$$\text{(ii)}\Rightarrow \varepsilon_r\sum\left[lA_l b^{l-1}-\left(l+1\right)\frac{\bar{B}_l}{b^{l+2}}\right]P_l\left(\cos\theta\right)=-E_0\cos\theta-\sum\left(l+1\right)\frac{B_l}{b^{l+2}}P_1 c\left(\cos\theta\right);$$

$$\text{(iii)} \Rightarrow A_1 a^l+\frac{\bar{B}_l}{a^{l+1}}=0\Rightarrow -a^{2l+1}A_l \ .$$

Đối với l #1 :

$$\text{(i)}\ -E_0 b+\frac{B_1}{b^2}=A_1 b-\frac{a^3 A_1}{b^2}\Rightarrow B_1-E_0 b^3=A_1 2\left(b^3-a^3\right);$$

$$\text{(ii)}\ \varepsilon_r\left(A_1+2\frac{a^3 A_1}{b^3}\right)=-E_0-2\frac{B_1}{b^3}\Rightarrow -2B_1-E_0 b^3=\varepsilon_r A_1\left(b^3+2a^3\right).$$

Vì vậy $-3E_0 b^3=A_1\left[2\left(b^3-a^3\right)+\varepsilon_r\left(b^3+2a^3\right)\right]$; $A_1=\dfrac{-3E_0}{2\left[1-\left(a/b\right)^3\right]+\varepsilon_r\left[1+2\left(a/b\right)^3\right]}$.

$$V_{med}=\frac{-3E_0}{2\left[1-\left(a/b\right)^3\right]+\varepsilon_r\left[1+2\left(a/b\right)^3\right]}\left(r-\frac{a^3}{r^2}\right)\cos\theta,$$

$$E\left(r,\theta\right)=-\nabla V_{med}=\boxed{\frac{3E_0}{2\left[1-\left(a/b\right)^3\right]+\varepsilon_r\left[1+2\left(a/b\right)^3\right]}\left\{\left(1+\frac{2a^3}{r^3}\right)\cos\theta\hat{r}-\left(1-\frac{a^3}{r^3}\right)\sin\theta\hat{\theta}\right\}}\ .$$

Bài tập 4.25

Có 4 điện tích liên quan : (i) q , (ii) xung quanh điện tích phân cực q , (iii) điện tích bề mặt σ_b trên mặt trên của điện môi thấp hơn, (iv) điện tích bề mặt $\sigma_b^{'}$ trên bề mặt thấp hơn của điện môi bên trên. Khi nhìn phương trình. 4.39 , điện tích biên (ii) bằng $q_p = -q\left(\chi_e^{'}/1+\chi_e^{'}\right)$, vì vậy điện tích (điểm) toàn phần tại (0 , 0 , d) bằng $q_l = q + q_p = q/\left(1+\chi_e^{'}\right) = q/\varepsilon_r^{'}$. Như trong ví dụ. 4.8,

(a) $\sigma_b = \varepsilon_0\chi_e\left[\frac{-1}{4\pi\varepsilon_0}\frac{qd/\varepsilon_r^{'}}{\left(r^2+d^2\right)^{\frac{3}{2}}} - \frac{\sigma_b}{2\varepsilon_0} - \frac{\sigma_b^{'}}{2\varepsilon_0}\right]$ (ở đây $\sigma_b = P.\hat{n} = +P_z = \varepsilon_0\chi_e E_z$);

(b) $\sigma_b^{'} = \varepsilon_0\chi_e^{'}\left[\frac{1}{4\pi\varepsilon_0}\frac{qd/\varepsilon_r^{'}}{\left(r^2+d^2\right)^{\frac{3}{2}}} - \frac{\sigma_b}{2\varepsilon_0} - \frac{\sigma_b^{'}}{2\varepsilon_0}\right]$ (ở đây $\sigma_b = -P_z = -\varepsilon_0\chi_e^{'}E_z$);

Giải cho σ_b , $\sigma_b^{'}$: đầu tiên chia cho χ_e và $\chi_e^{'}$ và viết gọn:

$$\frac{\sigma_b^{'}}{\chi_e^{'}} - \frac{\sigma_b}{\chi_e} = \frac{1}{2\pi}\frac{qd/\varepsilon_r^{'}}{\left(r^2+d^2\right)^{\frac{3}{2}}} \Rightarrow \sigma_b^{'} = \chi_e^{'}\left[\frac{\sigma_b}{\chi_e} + \frac{1}{2\pi}\frac{qd/\varepsilon_r^{'}}{\left(r^2+d^2\right)^{\frac{3}{2}}}\right].$$

Thế biểu thức này vào (a) và giải tìm σ_b , dùng $\varepsilon_r^{'} = 1+\chi_e^{'}$:

$\sigma_b = \frac{-1}{4\pi}\frac{qd/\varepsilon_r^{'}}{\left(r^2+d^2\right)^{\frac{3}{2}}}\chi_e\left(1+\chi_e^{'}\right) - \frac{\sigma_b}{2}\left(\chi_e+\chi_e^{'}\right)$, vì vậy

$$\boxed{\sigma_b = \frac{-1}{4\pi}\frac{qd}{\left(r^2+d^2\right)^{\frac{3}{2}}}\frac{\varepsilon_r\chi_e^{'}/\varepsilon_r^{'}}{\left[1+\left(\chi_e+\chi_e^{'}\right)/2\right]}.}$$

Điện tích biên tổng cộng bằng $\sigma_t = \sigma_b + \sigma_b^{'} = \frac{1}{4\pi}\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\frac{(\chi_e^{'} - \chi_e)}{\varepsilon_r^{'}\left[1+(\chi_e+\chi_e^{'})/2\right]}$ (Nó

sẽ biến mất khi $\chi_e^{'} = \chi_e$) . Điện tích biên tổng cộng là (so sánh Pt. 4.51) :

$q_t = \frac{(\chi_e^{'} - \chi_e)q}{2\varepsilon_r^{'}\left[1+(\chi_e+\chi_e^{'})/2\right]} = \boxed{\left(\frac{\varepsilon_r^{'} - \varepsilon_r}{\varepsilon_r^{'} + \varepsilon_r}\right)\frac{q}{\varepsilon_r^{'}}}$, và vì thế

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q/\varepsilon_r^{'}}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_t}{\sqrt{x^2+y^2+(z+d)^2}}\right\}}$$ (đối với z > 0)

Trong khi đó, bởi vì $\frac{q}{\varepsilon_r^{'}} + q_t = \frac{q}{\varepsilon_r^{'}}\left[1+\frac{\varepsilon_r^{'}-\varepsilon_r}{\varepsilon_r^{'}+\varepsilon_r}\right] = \frac{2q}{\varepsilon_r^{'}+\varepsilon_r}$,

$$\boxed{V(r) = \frac{1}{4\theta\varepsilon_0}\frac{\left[2q/(\varepsilon_r^{'}+\varepsilon_r)\right]}{\sqrt{x^2+y^2+(z-d)^2}}}$$ (đối với z < 0).

Bài tập 4.26

Từ ví dụ. 4.5:

$$D = \begin{cases} 0, \ r < a \\ \frac{Q}{4\pi r^2}\hat{r}, \ r > a \end{cases} \quad , \quad E = \begin{cases} 0, \ r < a \\ \frac{Q}{4\pi\varepsilon r^2}\hat{r}, \ a < r < b \\ \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}, \ r > b \end{cases}$$

$$W=\frac{1}{2}\int D.E dr=\frac{1}{2}\frac{Q}{4\pi}{}^{2}4\pi\left\{\frac{1}{\varepsilon}\int_a^b\frac{1}{r^2}\frac{1}{r^2}r^2dr+\frac{1}{\varepsilon_0}\int_b^\infty\frac{1}{r^2}dr\right\}=\frac{Q^2}{8\pi}\left\{\frac{1}{\varepsilon}\left(\frac{-1}{r}\right)\Bigg|_a^b+\frac{1}{\varepsilon_0}\left(\frac{-1}{r}\right)\Bigg|_b^\infty\right\}22$$

$$=\frac{Q^2}{8\pi\varepsilon_0}\left\{\frac{1}{1+\chi_e}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\}=\boxed{\frac{Q^2}{8\pi\varepsilon_0\ 1+\chi_e}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)}.$$

Bài tập 4.27

Dùng pt. 4.55 : $W=\frac{\varepsilon_0}{2}\int E^2 dr$. Từ ví dụ. 4.2 và Pt. 3.103 ,

$$E=\begin{cases}\frac{-1}{3\varepsilon_0}P\hat{z}, & r<R\\ \frac{R^3P}{3\varepsilon_0 r^3}\ 2\cos\theta\hat{r}+\sin\theta\hat{\theta}\ , & r>R\end{cases}$$, vì vậy

$$W_{r<R}=\frac{\varepsilon_0}{2}\left(\frac{P}{3\varepsilon_0}\right)^2\frac{4}{3}\pi R^3=\frac{2\pi}{27}\frac{P^2R^3}{\varepsilon_0}\ .$$

$$W_{tot}=\boxed{\frac{2\pi R^3P^2}{9\varepsilon_0}}$$

Đây là năng lượng tương tác tĩnh điện của cấu hình, nhưng nó không phải là " công toàn phần cần thiết để tập hợp hệ ," vì nó bỏ đi cơ năng lên quan đến sự phân cực phân tử.

Dùng Pt. 4.58 : $W=\frac{1}{2}\int D.E dr$. Đối với $r<R, D=\varepsilon_0E+D=-\frac{1}{3}P+P=-2\varepsilon_0E$, vì vậy $\frac{1}{2}D.E=-2\frac{\varepsilon_0}{2}E^2$, và bây giờ đóng góp này là $-2\left(\frac{2\pi}{27}\frac{P^2R^3}{\varepsilon_0}\right)=-\frac{4\pi}{27}\frac{R^3P^2}{\varepsilon_0}$, chính xác triệt tiêu số hạng bên ngoài . Kết luận: $\boxed{W_{tot}=0}$. Điều này không có gì lạ, vì

đạo hàm trong phần. 4.4.3 tính toán công được thực hiện trên điện tích tự do, và trong bài toán này không có điện tích tự do in sight. Tuy nhiên, bởi vì đây là điện môi phi tuyến, kết quả không thể được giải thích là “ công cần thiết để gắn kết hệ thống’’ – cái sau sẽ hoàn toàn phụ thuộc vào cách bạn gắn kết nó .

Bài tập 4.28

Đầu tiên tìm điện dung, như hàm theo h :

Phần không khí :
$$\left.\begin{aligned} & E=\frac{2\lambda}{4\pi\varepsilon_0 s}\Rightarrow V=\frac{2\lambda}{4\pi\varepsilon_0}\ln(b/a) \\ & D=\frac{2\lambda^{'}}{4\pi s}\Rightarrow E=\frac{2\lambda^{'}}{4\pi\varepsilon s}\Rightarrow V=\frac{2\lambda^{'}}{4\pi\varepsilon}\ln(b/a)\ , \end{aligned}\right\}\Rightarrow\frac{\lambda}{\varepsilon_0}=\frac{\lambda^{'}}{\varepsilon};\lambda^{'}=\frac{\varepsilon}{\varepsilon_0}\lambda=\varepsilon_r\lambda.$$

Phần dầu :

$Q=\lambda^{'}h+\lambda(l-h)=\varepsilon_r\lambda h-\lambda h+\lambda l=\lambda\left[(\varepsilon_r-1)h+l\right]=\lambda(\chi_e h+l)$, ở đây l là tổng chiều dài.

$$C=\frac{Q}{V}=\frac{\lambda(\chi_e h+l)}{2\lambda\ln(b/a)}4\pi\varepsilon_0=2\pi\varepsilon_0\frac{(\chi_e h+l)}{\ln(b/a)}$$

Đây là lực toàn phần hướng lên được cho bởi Pt. 4.64 : $F=\frac{1}{2}V^2\frac{dC}{dh}=\frac{1}{2}V^2\frac{2\pi\varepsilon_0\chi_e}{\ln(b/a)}$

Trọng lực hướng xuống bằng $F=mg=\rho\pi(b^2-a^2)gh$

Bài tập 4.29

(a) Eq. 4.5 $\Rightarrow F_2=(p_2\nabla)E_1=p_2\frac{\partial}{\partial y}(E_1)$;

Pt. 3.103 $E_1=\frac{p_1}{4\pi\varepsilon_0 r^3}\hat{\theta}=-\frac{p_1}{4\pi\varepsilon_0 y^3}\hat{z}$. Do đó

$E_2=-\frac{p_1p_2}{4\pi\varepsilon_0}\left[\frac{d}{dy}\left(\frac{1}{y^3}\right)\right]\hat{z}=\frac{3p_1p_2}{4\pi\varepsilon_0 y^4}$ or $\boxed{F_2=\frac{3p_1p_2}{4\pi\varepsilon_0 r^4}\hat{z}}$ (hướng lên)

Để tính F_1 , đặt p_2 ở gốc tọa độ , theo hướng z; thế thì p_1 nằm tại $-r\hat{z}$, và hướng

theo hướng $-\hat{y}$. Vì vậy $F_1 = \ p_1\nabla\ \ E_2 = -p_1 \frac{\partial E_2}{\partial y}\Big|_{x=y=0,z=-r}$; chúng ta cần E_2 như hàm theo x , y , và z .

Từ pt. 3.104 : $E_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[\frac{3\ \ p_2.r\ \ r}{r^2} - p\ \right]$, ở đây $r = x\hat{x} + y\hat{y} + z\hat{z}$, $p_2 = -p_2\hat{y}$, và vì thế $p_2.r = -p_2 y$

$$E_2 = \frac{p_2}{4\infty\varepsilon_0}\left[\frac{-3y\ \ x\hat{x}+y\hat{y}+z\hat{z}\ \ +\ \ x^2+y^2+z^2\ \ \hat{y}}{x^2+y^2+z^2\ \ ^{5/2}}\right] = \frac{p_2}{4\pi\varepsilon_0}\left[\frac{-3xy\hat{x}+\ \ x^2-2y^2+z^2\ \ \hat{y}-3yz\hat{z}}{x^2+y^2+z^2\ \ ^{5/2}}\right]$$

$$\frac{\partial E_2}{\partial y} = \frac{p_2}{4\pi\varepsilon_0}\left\{-\frac{5}{2}\frac{1}{r^7}2y\left[-3xy\hat{x}+\ \ x^2-2y^2+z^2\ \ \hat{y}-3yz\hat{z}\right]+\frac{1}{r^5}\ \ -3x\hat{x}-4y\hat{y}=3z\hat{z}\ \ \right\}$$

$$\frac{\partial E_2}{\partial y}\Big|_{0,0} = \frac{p_2}{4\pi\varepsilon_0}\frac{-3z}{r^5}\hat{z}; F_1 = -p_1\left(\frac{p_2}{4\pi\varepsilon_0}\frac{3r}{r^5}\hat{z}\right) = \boxed{-\frac{3p_1p_2}{4\pi\varepsilon_0 r^4}\hat{z}.}$$

Kết quả phù hợp với điện luật III Newton: F_1=-F_2 .

(c) Từ trang 165 , $N_2 = \ \ p_2\times E_1\ \ +\ \ r\times F_2$. Số hạng thứ nhất được tính trong bài tập. 4.5 ; số hạng thứ hai chúng ta nhận được từ (a) , dùng $r = r\hat{y}$:

$$p_2\times E_1 = \frac{p_1p_2}{4\pi\varepsilon_0 r^3}\ \ -\hat{x}\ \ ;\ r\times F_2 = \ \ r\hat{y}\ \ \times\left(\frac{3p_1p_2}{4\pi\varepsilon_0 r^4}\hat{z}\right) = \frac{3p_1p_2}{4\pi\varepsilon_0 r^3}\hat{x}\ ;\ \text{vì vậy}\ \boxed{N_2 = \frac{2p_1p_2}{4\pi\varepsilon_0 r^3}\hat{x}}$$

Cái này bằng và ngược dấu với momen p_1 do p_2 , đối với tâm của p_1 (xem bài tập. 4.5) .

Bài tập 4.30

Lực toàn phần ở bên phải (xem đồ thị). Chú ý rằng các đường sức trường phải phình ở bên phải, như được biểu diễn, vì E vuông góc với bề mặt của mỗi vật dẫn .

Bài tập 4.31

$P = kr = k\,(x\hat{x} + y\hat{y} + z\hat{z}) \Rightarrow \rho_b = -\nabla . P = -k\,(1+1+1) = \boxed{-3k}$.

Điện tích biên thể tích tổng cộng : $\boxed{Q_{vol} = -3ka^3}$

$\sigma_b = P.\hat{n}$. Ở bề mặt trên, $\hat{n} = \hat{z}, z = a/2$; so $\sigma_b = ka/2$. Rõ ràng , $\boxed{\sigma_b = ka/2}$ trên tất cả sáu mặt .

Điện tích biên bề mặt toàn phần : $\boxed{Q_{sorf} = 6\,(ka/2)\,a^2 = 3ka^3}$. Điện tích biên tổng cộng bằng không .

Bài tập 4.32

$\oint D.da = Q_{f_{enc}} \Rightarrow D = \dfrac{q}{4\pi r^2}\hat{r}; E = \dfrac{1}{\varepsilon}D = \boxed{\dfrac{q}{4\pi\varepsilon_0 (1+\chi_e)\, r^2}\dfrac{\hat{r}}{r^2}}; P = \varepsilon_0 \chi_e E = \boxed{\dfrac{q\chi_e}{4\pi (1+\chi_e)}\dfrac{\hat{r}}{r^2}}.$

$\rho_b = -\nabla . P = -\dfrac{q\chi_e}{4\pi (1+\chi_e)}\left(\nabla . \dfrac{\hat{r}}{r^2}\right) = \boxed{-q_1 \dfrac{\chi_e}{1+\chi_e}\delta^3(r)}$; $(Eq.1.99)$; $\sigma_b = P.\hat{r} = \boxed{\dfrac{q\chi_e}{4\pi (1+\chi_e) R^2}}$

$Q_{surf} = \sigma_b\,(4\pi R^2) = \boxed{q\dfrac{\chi_e}{1+\chi_e}}$. Sự bù điện tích âm ở tâm:

$\int \rho_b dr = -\dfrac{q\chi_e}{1+\chi_e}\int \delta^3(r)\, dr = -\dfrac{q\chi_e}{1+\chi_e}$.

Bài tập 4.33

E liên tục (Pt. 4.29) ; $D\perp$ liên tục (Pt. 4.26 , với $\sigma_f = 0$) . Vì vậy

$E_{x_1} = E_{x_2}; D_{y_1} = D_{y_2} \Rightarrow \varepsilon_1 E_{y_1} = \varepsilon_2 E_{y_2}$, và vì thế

$\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{E_{x_2}/E_{y_2}}{E_{x_1}/E_{y_1}} = \dfrac{E_{y_1}}{E_{y_2}} = \dfrac{\varepsilon_2}{\varepsilon_1}$. qed

Nếu 1 là không khí và 2 là điện môi , $\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{\varepsilon_2}{\varepsilon_1} > 1$, và các đường sức trường bị cong ra khỏi đường pháp tuyến . Đây là phía đối diện của các tia sáng, vì vậy thấu kính lồi sẽ làm lệch tiêu các đường sức trường.

Bài tập 4.34

Từ Pt. 4.39 , momen lưỡng cực toàn phần ở tâm là $p^{'} = p - \frac{1}{1+\chi_e} p = \frac{1}{1+\chi_e} p = \frac{1}{\varepsilon_r} p$. Chúng ta muốn thế được tạo ra bởi p' (tại tâm) và σ_b (tại R). Dùng phương pháp tách biến :

$$\left\{ \begin{array}{ll} Outside: V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) & Eq.3.72 \\ \\ Inside: V(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{\varepsilon_r r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta) & Eqs.3.66, 3.102 \end{array} \right\}$$

V liên tục tại R

$$R \Rightarrow \left\{ \begin{array}{lll} \frac{B_l}{R^{l+1}} = A_l R^l & & B_l = R^{2l+1} A_l \\ & \text{hoặc} & \\ \frac{B_1}{R^2} = \frac{1}{4\pi\varepsilon_0} \frac{p}{\varepsilon_r R^2} + A_1 B, & & B_1 = \frac{p}{4\pi\varepsilon_0\varepsilon_r} + A_1 R^3 \end{array} \right\}.$$

$$\left.\frac{\partial V}{\partial r}\right|_{R+} - \left.\frac{\partial V}{\partial r}\right|_{R-} = -\sum (l+1) \frac{B_l}{R^{l+2}} P_l(cos\theta) + \frac{1}{4\pi\varepsilon_0} \frac{2p\cos\theta}{\varepsilon_r R^3} - \sum l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\varepsilon_0}\sigma_b$$

$$= -\frac{1}{\varepsilon_0} P.\hat{r} = -\frac{1}{\varepsilon_0}(\varepsilon_0 \chi_e E \hat{r}) = \chi_e \left.\frac{\partial V}{\partial r}\right|_{R-} = \chi_e \left\{ -\frac{1}{4\pi\varepsilon_0} \frac{2p\cos\theta}{\varepsilon_r R^3} + \sum l A_l R^{l-1} P_l(\cos\theta) \right\}.$$

$$-(l+1)\frac{B_l}{R^{l+2}} - lA_l R^{l-1} = \chi_e l A_l R^{l-1} \ (l \# 1); or -(2l+1) A_l R^{l-1} = \chi_e l A_l R^{l-1} \Rightarrow A_l = 0 \ (l \# 1).$$

Đối với l = 1

$$-2\frac{B_1}{R^3} + \frac{1}{4\pi\varepsilon_0}\frac{2p}{\varepsilon_r R^3} - A_l = \chi_e \left(-\frac{1}{4\pi\varepsilon_0}\frac{2p}{\varepsilon_r R^3} + A_1 \right) - B_1 + \frac{p}{4\pi\varepsilon_0\varepsilon_r} - \frac{A_1 R^3}{2} = -\frac{1}{4\pi\varepsilon_0}\frac{\chi_e p}{\varepsilon_r} + \chi_e \frac{A_1 R^3}{2}$$

$$-\frac{p}{4\pi\varepsilon_0\varepsilon_r} - A_1 R^3 + \frac{p}{4\pi\varepsilon_0\varepsilon_r} - \frac{A_1 R^3}{2} = -\frac{1}{4\pi\varepsilon_0}\frac{\chi_e p}{\varepsilon_r} + \chi_e \frac{A_1 R^3}{2} \Rightarrow \frac{A_1 R^3}{2}(3+\chi_e) = \frac{1}{4\pi\varepsilon_0}\frac{\chi_e p}{\varepsilon_r}$$

$$\Rightarrow A_1 = \frac{1}{4\pi\varepsilon_0}\frac{2\chi_e p}{R^3 \varepsilon_r (3+\chi_e)} = \frac{1}{4\pi\varepsilon_0}\frac{2(\varepsilon_r - 1)p}{R^3\varepsilon_r(\varepsilon_r+2)}; B_1 = \frac{p}{4\pi\varepsilon_0\varepsilon_r}\left[1 + \frac{2(\varepsilon_r - 1)}{(\varepsilon_r+2)}\right] = \frac{p}{4\pi\varepsilon_0\varepsilon_r}\frac{3\varepsilon_r}{\varepsilon_r+2}.$$

$$\boxed{V(r,\theta) = \left(\frac{q\cos\theta}{4\pi\varepsilon_0 r^2}\right)\left(\frac{3}{\varepsilon_r+2}\right)(r \geq R)}\ .$$

$$r \leq R, V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{\varepsilon_r r^2} + \frac{1}{4\pi\varepsilon_0}\frac{pr\cos\theta}{R^3}\frac{2(\varepsilon_r - 1)}{\varepsilon_r + 2}$$

Trong khi đó , đối với $\boxed{\frac{p\cos\theta}{4\pi\varepsilon_0 r^2\varepsilon_r}\left[1+2\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)\frac{r^3}{R^3}\right](r \leq R)\ .}$

Bài tập 4.35

Hai nghiệm cho trước,V_1 (and $E_1 = -\nabla V_1, D_1 = \varepsilon E_1$) V_2 ($E_2 = -\nabla V_2, D_2 = \varepsilon E_2$) , xác định $V_3 \equiv V_2 - V_1$ ($E_3 = E_2 - E_1, D_3 = D_2 - D_1$) .

$\int_v \nabla.(V_3 D_3)dr = \int_s V_3 D_3.da = 0,$ ($V_3 = 0$ trên S) , vì vậy $\int(\nabla V_3).D_3 dr + \int V_3(\nabla.D_3)dr = 0.$

Nhưng $\nabla.D_3 = \nabla.D_2 - \nabla D_1 = \rho_f - \rho_f = 0$, và $\nabla.V_3 = \nabla.V_2 - \nabla V_1 = -E_2 + E_1 = -E_3$, vì vậy $\int E_3.D_3 dr = 0$.

Nhưng $D_3 = D_2 - D_1 = \varepsilon E_2 - \varepsilon E_1 = \varepsilon E_3$, vì vậy $\int\varepsilon(E_3)^2 dr = 0$. Nhưng $\varepsilon > 0$, vì vậy $E_3 = 0$, vì vậy V_2-V_1=constant . Nhưng tại bề mặt , V_2=V_1 , so V_2=V_1 ở mọi nơi. qed

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Bài tập 4.36

(a) Thế được đề xuất : $\boxed{V(r) = V_0\frac{R}{r}}$ Nếu vậy , thì $\boxed{E = -\nabla V = V_0\frac{R}{r^2}\hat{r}}$ trong trường hợp mà $\boxed{P = \varepsilon_0\chi_e V_0\frac{R}{r^2}\hat{r},}$ trong vùng z < 0 . (Tất nhiên, P = 0 đối với z > 0) . Do đó $\sigma_b = \varepsilon_0\chi_e V_0\frac{R}{r^2}(\hat{r}.\hat{n}) = \boxed{-\frac{\varepsilon_0\chi_e V_0}{R}.}$ (Chú ý : hướng ra ngoài điện môi $\Rightarrow \hat{n} = -\hat{r}$) . Cái này ở trên bề mặt tại r = R . Mặt phẳng z = 0 không mang điện tích biên , bởi vì $\hat{n} = \hat{z} \perp \hat{r}$. Cũng không có điện tích biên thể tích (Pt. 4.39) . Nếu V có đối xứng cầu cần thiết, điện tích toàn phần phải đều:

$\sigma_{tot} 4\pi R^2 = Q_{tot} = 4\pi\varepsilon_0 R V_0$ (bởi vì $V_0 = Q_{tot}/4\pi\varepsilon_0 R$, so $\sigma_{tot} = \varepsilon_0 V_0/R$. Do đó

$$\boxed{\sigma_f = \begin{Bmatrix} \varepsilon_0 V_0/R & \text{on northern hemisphere} \\ \varepsilon_0 V_0/R \quad 1+\chi_e & \text{on southern hemisphere} \end{Bmatrix}.}$$

(b) Bằng cách xây dựng, $\sigma_{tot} = \sigma_b + \sigma_f = \varepsilon_0 V_0/R$ là đều (trên bán cầu bắc $\sigma_b = 0, \sigma_f = \varepsilon_0 V_0/R$; trên bán cầu nam $\sigma_b = -\varepsilon_0 \chi_e V_0/R$, so $\sigma_f = \varepsilon V_0/R$) . Thế của quả cầu điện tích đều là

$$V_0 = \frac{Q_{tot}}{4\pi\varepsilon_0 r} = \frac{\sigma_{tot} \quad 4\pi R^2}{4\pi\varepsilon_0 r} = \frac{\varepsilon_0 V_0}{R}\frac{R^2}{\varepsilon_0 r} = V_0 \frac{R}{r} \ .$$

(c) Bởi vì mọi thứ phù hợp, và các điều kiện biên ($V = V_0$ at r = R at ∞) phù hợp, Bài tập. 4.35 đảm bảo rằng đây là nghiệm.

(d) Hình (b) làm việc theo cách, nhưng hình. (a) thì không : ở bề mặt phẳng, P không vuông góc với, vì vậy chúng ta nhận được điện tích biên trên bề mặt này, phá vỡ đối xứng.

Bài tập 4.37

$E_{ext} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$ Bởi vì hình cầu nhỏ, cái này về cơ bản là không đổi, và vì thế

$P = \frac{\varepsilon_0 \chi_e}{1+\chi_e/3} E_{ext}$ (Ví dụ. 4.7)

$$F = \int \left(\frac{\varepsilon_0 \chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\varepsilon_0 s}\right)\frac{d}{ds}\left(\frac{\lambda}{2\pi\varepsilon_0 s}\right)\hat{s}dr = \left(\frac{\varepsilon_0 \chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda}{2\pi\varepsilon_0 s}\right)^2 \left(\frac{1}{s}\right)\left(\frac{-1}{s^2}\right)\hat{s}\int dr$$

$$= \left(\frac{-\chi_e}{1+\chi_e/3}\right)\left(\frac{\lambda^2}{4\pi^2\varepsilon_0}\right)\frac{1}{s^3}\frac{4}{3}\pi R^3 \hat{s} = \boxed{-\left(\frac{\chi_e}{3+\chi_e}\right)\frac{\lambda^2 R^3}{\pi\varepsilon_0 s^3}\hat{s}.}$$

Bài tập 4.38

Mật độ nguyên tử bằng $N = \frac{1}{4/3 \ \pi R^3}$. Điện trường vĩ mô E bằng , ở đây E_{self} là trường trung bình trên hình cầu do chính nguyên tử .

$p = \alpha E_{else} \Rightarrow P = N\alpha E_{else}$.

[Thực sự, nó là trường tại tâm, không phải trung bình trên hình cầu, mà thuộc ở đây, nhưng quả thực hai cái bằng nhau, như chúng ta thấy trong bài tập. 3.41.]
Bây giờ

(Pt. 3.105) , vì vậy $E_{self} = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{\alpha}{R^3}E_{esle}$

$$E = -\frac{1}{4\pi\varepsilon_0}\frac{\alpha}{R^3}E_{esle} + E_{else} = \left(1 - \frac{\alpha}{4\pi\varepsilon_0 R^3}\right)E_{esle} = \left(1 - \frac{N\alpha}{3\varepsilon_0}\right)E_{esle} \ .$$

Vì vậy

$$P = \frac{N\alpha}{1 - N\alpha/3\varepsilon_0}E = \varepsilon_0\chi_e E,$$

Và vì thế

$$\chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0}.$$

Giải tìm α : $\chi_e - \dfrac{N\alpha}{3\varepsilon_0}\chi_e = \dfrac{N\alpha}{\varepsilon_0} \Rightarrow \dfrac{N\alpha}{\varepsilon_0}\left(1 + \dfrac{\chi_e}{3}\right) = \chi_e,$

Hoặc $\alpha = \dfrac{\varepsilon_0}{N}\dfrac{\chi_e}{1 + \chi_e/3} = \dfrac{3\varepsilon_0}{N}\dfrac{\chi_e}{3 + \chi_e}$. Nhưng $\chi_e = \varepsilon_r - 1$, vì vậy $\alpha = \dfrac{3\varepsilon_0}{N}\left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$. qed

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Bài tập 4.39

Đối với khí lí tưởng, N = số Avagadro / 22.4 lít

$= 6.02\times10^{23} \Big/ 22.4\times10^{-3} = 2.7\times10^{25}. N\alpha/\varepsilon_0 = 2.7\times10^{25} \quad 4\pi\varepsilon_0 \times 10^{-30} \quad \beta/\varepsilon_0 = 3.4\times10^{-4}\beta$

, ở đây là số được liệt kê trong bảng 4.1 .

$$\left.\begin{array}{l}
\text{H} : \beta = 0.667, N\alpha\Big/\varepsilon_0 = \ 3.4\times10^{-4} \quad 0.67 \ = \ 2.3\times10^{-4} \ , \chi_e = \ 2.5\times10^{-4} \\
\text{He} : \beta = 0.205, Na\Big/\varepsilon_0 = \ 3.4\times10^{-4} \quad 0.21 \ = \ 7.1\times10^{-5} \ , \chi_e = \ 6.5\times10^{-4} \\
\text{Ne} : \beta = 0.396, N\alpha\Big/\varepsilon_0 = \ 3.4\times10^{-4} \quad 0.40 \ = 1.4\times10^{-4} \ , \chi_e = 1.3\times10^{-4} \\
\text{Ar} : \beta = 1.64, N\alpha\Big/\varepsilon_0 = \ 3.4\times10^{-4} \quad 1.64 \ = 5.6\times10^{-4} \ , \chi_e = 5.2\times10^{-4}
\end{array}\right\} \text{phù hợp khá tốt .}$$

Bài tập 4.40

(a)

$$\langle u\rangle = \frac{\int_{-pE}^{pE} ue^{-u/kT}du}{\int_{-pE}^{pE} e^{-u/kT}} = \frac{kT\ ^2 e^{-u/kT}\left[-\ u/kT\ \ -1\right]\Big|_{-pE}^{pE}}{-kTe^{-u/kT}\Big|_{-pE}^{pE}}$$

$$= kT\left\{\frac{\left[e^{-pE/kT}-e^{pE/kT}\right]+\left[\ pE/kT\ \ e^{-pE/kT}+\ \ pE/kT\ \ e^{pE/kT}\right]}{e^{-pE/kT}-e^{pE/kT}}\right\}$$

$$= kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth\left(\frac{pE}{kT}\right).$$

$$P = N\ \ p\ \ ; p = \ \ p\cos\theta\ \ \hat{E} = \ \ P.E\ \ \ \hat{E}/E\ \ = -\langle u\rangle\ \ \hat{E}/E\ \ ; P = N_p\frac{-\langle u\rangle}{pE} = \boxed{N_p\left\{\coth\left(\frac{pE}{kT}\right)-\frac{kT}{pE}\right\}}.$$

Đặt $y \equiv P/N\ \ p\ \ ; x \equiv pE/kT$. Thế thì $y = \coth x - 1/x$.

Khi $x \to 0, y = \left(\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+...\right)-\frac{1}{x} = \frac{x}{3}-\frac{x^3}{45}+... \to 0$, vì vậy đồ thị bắt đầu tại gốc tọa độ, với hệ số góc ban đầu bằng $1/3$. Khi $x \to \infty, y \to \coth\ \ \infty\ \ = 1$, vì vậy đồ thị tiến tới tiệm cận y = 1 (xem hình) .

(b) Đối với $x, y \approx \frac{1}{3}x,$ nhỏ, vậy $\frac{P}{N_p} \approx \frac{pE}{3kT}$, or $P \approx \frac{Np^2}{3kT}E = \varepsilon_0\chi_e E \Rightarrow P$ tỉ lệ với E , và

$$\boxed{\chi_e = \frac{Np^2}{3\varepsilon_0 kT}}$$

Đối với nước tại

$$20^0 = 293K, p = 6.1\times10^{-30}Cm; N = \frac{molecules}{volume} = \frac{molecules}{mole}\times\frac{moles}{gram}\times\frac{gram}{volume}.$$

$$N = \ \ 6.0\times10^{23}\ \ \times\left(\frac{1}{18}\right)\times\ \ 10^6\ \ = 0.33\times10^{29}; \chi_e = \frac{0.33\times10^{29}\ \ \ 6.1\times10^{-30}\ \ ^2}{3\ \ 8.85\times10^{-12}\ \ 1.38\times10^{-23}\ \ \ 293} = \boxed{12.}$$

Table 4.2 gives an experimental value of 79 , so it's pretty far off .
For water vapor at $100^o = 373K$, treated as an ideal gas ,

$$\frac{volume}{mole} = \left(22.4\times10^{-3}\right)\times\left(\frac{373}{293}\right) = 2.85\times10^{-2} m^3$$

$$N = \frac{6.0\times10^{23}}{2.85\times10^{-2}} = 2.11\times10^{25}; \chi_e = \frac{\left(2.11\times10^{25}\right)\left(6.1\times10^{-30}\right)^2}{3\left(8.85\times10^{-12}\right)\left(1.38\times10^{-23}\right)\left(373\right)} = \boxed{5.7\times10^{-3}.}$$

Bảng 4.2 cho 5.9×10^{-3} , vì vậy lúc này phù hợp khá tốt.

Chương 5
Tĩnh từ

Bài tập 5.1

Bởi vì $v\times B$ hướng lên, và đó cũng là hướng của lực, q phải dương. Để tìm R , theo a và d , dùng định lí Pythago :

$$\left(R-d\right)^2 + a^2 = R^2 \Rightarrow R^2 - 2Rd + d^2 + a^2 = R^2 \Rightarrow R = \frac{a^2+d^2}{2d}$$

Công thức cyclotron cho ta

$$p = qBR = \boxed{qB\frac{a^2+d^2}{2d}} .$$

Bài tập 5.2

Nghiệm tổng quát là (Eq. 5.6) :

$$y\left(t\right) = C_1\cos\left(\omega t\right) + C_2\sin\left(\omega t\right) + \frac{E}{B}t + C_3; z\left(t\right) = C_2\cos\left(\omega t\right) - C_1\sin\left(\omega t\right) + C_4$$

(a) $y\left(0\right) = z\left(0\right) = 0; \dot{y}\left(0\right) = E/B; \dot{z}\left(0\right)$. Dùng những cái này để xác định C_1 , C_2 , C_3 , và C_4 .

$y\left(0\right) = 0 \Rightarrow C_1 + C_3 = 0; \dot{y}\left(0\right) = \omega C_2 + E/B \Rightarrow C_2 = 0; z\left(0\right) = 0 \Rightarrow C_2 + C_4 = 0 \Rightarrow C_4 = 0$,

$\dot{z}\left(0\right) = 0 \Rightarrow C_1 = 0$ và vì thế $C_3=0$. Vì vậy $\boxed{y\left(t\right) = Et/B; z\left(t\right) = 0}$. Điều này có ý nghĩa gì ? Lực từ bằng $q\left(v\times B\right) = -q\left(E/B\right)B\hat{z} = -qE$, nó triệt tiêu lực điện ; vì không có lực toàn phần, hạt di chuyển theo đường thẳng với tốc độ không đổi.

(b) Giả sử nó bắt đầu di chuyển từ gốc tọa độ, vì vậy $C_3 = -C_1$, $C_4 = -C_2$
chúng ta có $\dot{z}(0) = 0 \Rightarrow C_1 = 0 \Rightarrow C_3 = 0;$

$$\dot{y}(0) = -\frac{E}{2B} \Rightarrow C_2\omega + \frac{E}{B} = \frac{E}{2B} \Rightarrow C_2 = -\frac{E}{2\omega B} = -C_4;\ y(t) = -\frac{E}{2\omega B}\sin(\omega t) + \frac{E}{B}t;$$

$$z(t) = -\frac{E}{2\omega B}\cos(\omega t) + \frac{E}{2\omega B}$$, hoặc

$$\boxed{y(t) = \frac{E}{2\omega B}\left[2\omega t - \sin(\omega t)\right];\ z(t) = \frac{E}{2\omega B}\left[1 - \cos(\omega t)\right].}$$. Đặt $\beta \equiv E/2\omega B$

Thế thì

$$y(t) = \beta\left[2\omega t - \sin(\omega t)\right];\ z(t) = \beta\left[1 - \cos(\omega t)\right];\ (y - 2\beta\omega t) = -\beta\sin(\omega t),\ (z - \beta) = -\beta\cos(\omega t) \Rightarrow$$

.

$(y - 2\beta\omega t)^2 + (z - \beta)^2 = \beta^2$. Đây là đường tròn mà tâm của bán kính của nó chuyển động sang phải với tốc độ không đổi: $y_0 = 2\beta\omega t;\ z_0 = \beta$.

(c)

$$\dot{z}(0) = \dot{y}(0) = \frac{E}{B} \Rightarrow -C_1\omega = \frac{E}{B} \Rightarrow C_1 = -C_3 = -\frac{E}{\omega B};\ C_2\omega + \frac{E}{B} = \frac{E}{B} \Rightarrow C_2 = C_4 = 0$$

$$y(t) = -\frac{E}{\omega B}\cos(-\omega t) + \frac{E}{B}t + \frac{E}{\omega B};\ z(t) = \frac{E}{\omega B}\sin(\omega t).\ \boxed{y(t) = \frac{E}{\omega B}\left[t + \omega t - \cos(\omega t)\right];\ z(t) = \frac{E}{\omega B}\sin(\omega t).}$$

Đặt $\beta \equiv E/\omega B$, thế thì

$\left[y - \beta(1 + \omega t)\right] = -\beta\cos(\omega t),\ z = \beta\sin(\omega t);\ \left[y - \beta(1 + \omega t)\right]^2 + z^2 = \beta^2$. Đây là đường tròn bán kính β tâm của nó ở tại $y_0 = \beta(1 + \omega t),\ z_0 = 0$.

Bài tập 5.3

(a) Từ Pt. 5.2 , $F = q\left[E + (v \times B)\right] = 0 \Rightarrow E = vB \Rightarrow \boxed{v = \frac{E}{B}}$

(b) Từ pt. 5.3 , $mv = qER \Rightarrow \frac{q}{m} = \frac{v}{BR} = \boxed{\frac{E}{B^2R}}$.

Bài tập 5.4

Giả sử rằng I chảy ngược chiều kim đồng hồ (nếu không, đổi dấu trong đáp số) . Lực ở vế trái (hướng sang trái) triệt tiêu lực ở phía phải (hướng sang bên phải) ; lực ở trên bằng $IaB = Iak\ a/2 = Ika^2/2$, (hướng lên trên) , và lực ở dưới bằng $IaB = -I\sqrt{a^2+b^2}ka^2/2$ (cũng hướng lên trên) . Vì vậy lực toàn phần bằng $F = \boxed{Ika^2\hat{z}.}$

Bài tập 5.5

(a) $\boxed{K = \frac{I}{2\pi a}}$ vì chiều dài vuông góc với dòng chảy là đường tròn.

(b) $J = \frac{\alpha}{s} \Rightarrow I = \int Jda = \alpha\int\frac{1}{s}sdsd\phi = 2\pi\alpha\int ds = 2\pi\alpha a \Rightarrow \alpha = \frac{I}{2\pi a}; J = \boxed{\frac{I}{2\pi as}}$

Bài tập 5.6

(a) $v = \omega r$, so $\boxed{K = \sigma\omega r}$ (b) $v = \omega r\sin\theta\phi \Rightarrow \boxed{J = \rho\omega r\sin\theta\phi}$ ở đây $\rho \equiv Q/\ 4/3\ \pi R^3$

Bài tập 5.7

$\frac{dP}{dt} = \frac{d}{dt}\int_v \rho r.dr = \int\left(\frac{\partial\rho}{\partial t}\right)rdr = -\int\ \nabla.J\ \ rdr$ (bằng phương trình liên tục) . Bây giờ quy tắc tích #5 cho ta $\nabla.\ xJ\ = x\ \nabla.J\ + J.\ \nabla x$. Nhưng $\nabla x = \hat{x}$, vì vậy $\nabla.\ xJ\ = x\ \nabla.J\ + J_z$.

Vì thế $\int_v \nabla.J\ \ xdr = \int_v \nabla.\ xJ\ \ dr - \int_v J_x dr$

Số hạng đầu tiên bằng $\int_s xJ.da$ (qua định lí divergence) , và bởi vì J hoàn toàn bên trong , nó bằng không trên bề mặt. Do đó $\int_v \nabla.J\ \ xdr = -\int_v J_x dr$, hoặc , kết hợp cái này với các thành phần y và z , $\int_v \nabla.J\ \ rdr = -\int_v Jdr$. Hoặc quay lại dòng thứ nhất, $\frac{dP}{dt} = \int Jdr$ qed

Bài tập 5.8

(a) Dùng Pt. 5.35 , với $z = R, \theta_2 = -\theta_1 = 45^0$, và bốn phía : $B = \boxed{\frac{\sqrt{2}\mu_0 I}{\pi R}.}$

(b) $z = R, \theta_2 = -\theta_1 = \frac{\pi}{n}$, và n phía: $B = \boxed{\frac{n\mu_0 I}{2\pi R} \sin \ \pi/n \ .}$

(c) Đối với nhỏ $\theta, \sin\theta \approx \theta$. Khi $n \to \infty, B \to \frac{n\mu_0 I}{2\pi R}\left(\frac{\pi}{n}\right) = \boxed{\frac{\mu_0 I}{2R}}$ (tương tự như Pt. 5.38 , với z = 0)

Bài tập 5.9

(a) Các phần đường thẳng không tạo ra trường tại P . Hai phần tư đường tròn cho $B = \boxed{\frac{\mu_0 I}{s}\left(\frac{1}{a} - \frac{1}{b}\right)}$ (out) .

(b) Hai phần hai đường thẳng tương tự như một đường thẳng không xác định: $\frac{\mu_0 I}{2\pi R}$; Nửa đường tròn đóng góp $\frac{\mu_0 I}{4R}$

Vì vậy $B = \boxed{\frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)}$ (vào trang giấy)

Bài tập 5.10

(a) Các lực ở hai phía triệt tiêu . Ở dưới, $B = \frac{\mu_0 I}{2\pi s} \Rightarrow F = \left(\frac{\mu_0 I}{2\pi s}\right) Ia = \frac{\mu_0 I}{2\pi y}\hat{z};$ (lêm) . Ở trên, $B = \frac{\mu_0 I}{2\pi \ s + a} \Rightarrow F = \frac{\mu_0 I^2 a}{2\pi \ s + a}$ (xuống) . Lực toàn phần bằng $\boxed{\frac{\mu_0 I^2 a^2}{2\pi s \ s + a}}$ (lên) .

(b) Lực ở dưới giống như trước, $\mu_0 I^2 / 2\pi$. Ở phía trái , $B = \frac{\mu_0 I}{2\pi y}\hat{z}$

$dF = I \;\; dl \times B \;\; = I \;\; dx\hat{x} + dy\hat{y} + dz\hat{z} \;\; \times \left(\frac{\mu_0 I}{2\pi y} \hat{z} \right) = \frac{\mu_0 I^2}{2\pi y} \;\; -dx\hat{y} + dy\hat{x}$. Nhưng thành phần x triệt tiêu số hạng tương ứng ở vế phải, và $F_y = -\frac{\mu_0 I^2}{2\pi} \int_{s/\sqrt{3}}^{s/\sqrt{3}+a/2} \frac{1}{y} dx$. Ở đây $y = \sqrt{3}x$, vì vậy

$F_y = -\frac{\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(\frac{s/\sqrt{3} + a/2}{s/\sqrt{3}} \right) = -\frac{\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(1 + \frac{\sqrt{3}a}{2s} \right)$. Lực ở vế phải tương tự , vì vậy lực toàn phần trên tam giác là $\boxed{\frac{\mu_0 I^2}{2\pi}\left[1 - \frac{2}{\sqrt{3}} \ln\left(1 + \frac{\sqrt{3}a}{2s} \right) \right]}$

Bài tập 5.11

Dùng Pt. 5.38 đối với vòng có độ rộng dz , với $I \to nIdz$

$B = \frac{\mu_0 nI}{2} \int \frac{a^2}{a^2 + z^2 \;\; ^{3/2}} dz$. Nhưng $z = a\cot\theta$

Vì vậy $dz = -\frac{a}{\sin^2\theta}$, và $\frac{1}{a^2 + z^2 \;\; ^{3/2}} = \frac{\sin^3\theta}{a^3}$

Vì vậy

$B = \frac{\mu_0 nI}{2} \int \frac{a^2 \sin^3\theta}{a^3 \sin^2\theta} \;\; -ad\theta \;\; = -\frac{\mu_0 nI}{2} \int \sin\theta d\theta = \frac{\mu_0 nI}{2} \cos\theta \Big|_{\theta_1}^{\theta_2} = \boxed{\frac{\mu_0 nI}{2} \;\; \cos\theta_2 - \cos\theta_1 \;\; .}$

Đối với solenoid vô hạn , $\theta_2 = 0, \theta_1 = \pi$, vì vậy $\cos\theta_2 - \cos\theta_1 \;\; = 1 - \;\; -1 \;\; = 2$

và $B = \boxed{\mu_0 nI.}$

Bài tập 5.12

Lực hút từ trên một đơn vị chiều dài (Pt. 5.37 và 5.13) : $f_m = \frac{\mu_0}{2\pi} \frac{\lambda^2 v^2}{d}$.

Điện trường của một sợi dây (Pt. 2.9) : $E = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{s}$. Lực đẩy điện trên một đơn vị chiều dài trên các dây kia $f_e = \frac{1}{2\pi\varepsilon_0} \frac{\lambda^2}{d}$. Chúng cân bằng khi $\mu_0 v^2 = \frac{1}{\varepsilon_0}$, hoặc

$\boxed{v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$. Thế số vào $v = \frac{1}{\sqrt{8.85\times10^{-12} \quad 4\pi\times10^{-7}}} = \boxed{3.00\times10^{8}\ m/s}$

Đây đúng là tốc độ ánh sáng (!) , vì vậy thực sự bạn có thể không bao giờ có được dây đi đủ nhanh ; lực điện luôn luôn chiếm ưu thế.

Bài tập 5.13

(a) $\oint B.dl = B2\pi s = \mu_0 I_{enc} \Rightarrow \boxed{B = \begin{cases} 0, & \text{For s < a} \\ \frac{\mu_0 I}{2\pi s}\hat{\phi}, & \text{For s > a} \end{cases}}$

(b) $j = ks; I = \int_0^a J da = \int_0^a ks \ 2\pi s \ ds = \frac{2\pi k a^3}{3} \Rightarrow k = \frac{3I}{2\pi a^3}$.

$\frac{2\pi k s^3}{3} = I\frac{s^3}{a^3}$, đối với $s < a; I_{enc} = I$, đối với $s > a$. Vì vậy $B = \boxed{\begin{cases} \frac{\mu_0 I s^2}{2\pi a^3}\phi, \text{For s < a} \\ \frac{\mu_0 I}{2\pi s}\phi, \text{For s > a} \end{cases}}$

Problem 5.14

By the right-hand-rule , the field points in the $-\hat{y}$ direction for z > 0 , and in the $+\hat{y}$ direction for z < 0 . At z = 0 , B = 0 . Use the amperian loop shown :

$\oint B.dl = Bl = \mu_0 I_{enc} = \mu_0 lzJ \Rightarrow \boxed{B = -\mu_0 Jz\hat{y}}$ $-a < z < a$. If z > a , $I_{enc} = \mu_0 laJ$,

So $\boxed{B = \left\{ \begin{cases} -\mu_0 Ja\hat{y}, \text{for z > + a} \\ +\mu_0 Ja\hat{y}, \text{for z > -a} \end{cases} \right\}}$

Problem 5.15

The field inside a solenoid is $\mu_0 nI$, and outside it is zero . The outer solenois's field points to the left $-\hat{z}$ whereas the inner one points to the right $+\hat{z}$. So :(i)

$\boxed{B = \mu_0 I \ n_1 - n_2 \ \hat{z},}$ (ii) $\boxed{B = -\mu_0 I n_2 \hat{z}}$ (iii) $\boxed{B = 0}$.

Problem 5.16

From Ex. 5.8 , the top plate produces a field $\mu_0 K/2$ (aiming out of the page , for points above it , and into the page , for points below) . The bottom plate produces a field $\mu_0 K/2$ (aiming into yhe page , for points above it , and out of the page , for points below) . Above and below both plates the two fields cancel ; between the plates they add up to $\mu_0 K$, pointing in .

(a) $\boxed{B = \mu_0 \sigma v \ \ in}$ betweem the plates , $\boxed{B=0}$ elsewhere .

(b) The Lorentz Force law says $F = \int K \times B \ \ da$, so the force per unit area is $f = K \times B$. Here $K = \sigma v$, to the right , and B (the field of the lower plate) is , into the pate . So $\boxed{f_m = \mu_0 \sigma^2 v^2 / 2 \ \ up \ .}$

(c) The electric field of the lower plate is $\sigma / 2\varepsilon_0$; the electric force per unit area on the upper plate is $\boxed{f_e = \sigma^2 / 2\varepsilon_0 (down)}$. They balance if

$\mu_0 v^2 = 1 / \varepsilon_0 , or \boxed{v = 1 / \sqrt{\varepsilon_0 \mu_0} = c}$ (the speed of light) , as in Prob. 5.12.

Problem 5.17

We might as well orient the axes so the field point r lies on the y axis : x=(0 , y . 0) . Consider a source point at (x' , y' , z') on loop # 1 :

$$= -x'\hat{x} + \ y - y' \ \ \hat{y} - z'\hat{z}; dl' = dx'\hat{x} + dy'\hat{y};$$

$$dl' \times r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & dy' & 0 \\ -x' & y - y' & -z' \end{vmatrix} = \ -z'dy' \ \ \hat{x} + \ z'dx' \ \ \hat{y} + \left[\ y - y' \ \ dx' + x'dy' \right] \hat{z}.$$

$$dB_1 = \frac{\mu_0 I}{4\pi} \frac{dl' \times r}{r^3} = \frac{\mu_0 I}{4\pi} \frac{-z'dy' \ \ \hat{x} + \ z'dx' \ \ \hat{y} + \left[\ y - y' \ \ dx' + x'dy' \right] \hat{z}}{\left[\ x'^{\ 2} + \ y - y'^{\ 2} + \ z'^{\ 2} \right]^{3/2}}$$

Now consider the symmetrically placed source element on loop # 2 , at (x' , y' ,-z') . Since z' changes sign , while every – thing else is the same , the $\hat{x}$ and $\hat{y}$ components from dB_1 and dB_2 cancel , leaving only a $\hat{z}$ compoment . qed

With this , Ampere's law yields immediately ;

$$\boxed{B = \begin{cases} \mu_0 nI\hat{z}, & \text{Inside the solenoid;} \\ 0, & \text{Outside} \end{cases}}$$

(the same as for a circular solenoid – Ex.5.9).
For the toroid , $N/2\pi s = n$ (the number of turns per unit length) , so Eq.5.58 yields $B = \mu_0 nI$ inside , and zero outside , consident with the solenoid . [Note : $N/2\pi s = n$ applies only if the toroid is large in circumference , so that s is essentially constant over the cross – section .]

Problem 5.18

$\boxed{\text{It doesn't matter}}$. According to Theorem 2 , in Sect . 1.6.2 , $\int J.da$ is independent of surface , for any given boundary line , provided that J is divergenceless , which it is , for steady currents (Eq. 5.31) .

Problem 5.19

(a) $\rho = \dfrac{charge}{volume} = \dfrac{charge}{atom} \cdot \dfrac{atoms}{mole} \cdot \dfrac{moles}{gram} \cdot \dfrac{grams}{volume} = e \quad N \left(\dfrac{1}{M}\right) d$, where

e = charge of electron	$= 1.6\times10^{-19} C$
N = Avogadro's number	$= 6.0\times10^{23} mole$
M = atomic mass of copper	$= 64\, gm/mole$
d = density of copper	$= 9.0\, gm/cm^3$

$$\rho = 1.6\times10^{-19} \quad 6.0\times10^{23} \left(\frac{9.0}{64}\right) = \boxed{1.4\times10^4\, C/cm^3.}$$

(b) $J = \dfrac{I}{\pi s^2} = \rho v \Rightarrow v = \dfrac{I}{\pi s^2 \rho} = \dfrac{1}{\pi \;\; 2.5\times10^{-3} \;\; 1.4\times10^4} = \boxed{9.1\times10^{-3}\, cm/s}$ or about 33 cm/s . This is astonishing small – literally slower than a snail's pace .

(c) From Eq. 5.37 , $f_m = \dfrac{\mu_0}{2\pi}\left(\dfrac{I_1 I_2}{d}\right) = \dfrac{4\pi\times10^{-7}}{2\pi} = \boxed{2\times10^{-7}\, N/cm.}$

(d) , where $E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{d}$;

$$f_e = \frac{1}{2\pi\varepsilon_0}\left(\frac{\lambda_1\lambda_2}{d}\right) = \frac{1}{v^2}\frac{1}{2\pi\varepsilon_0}\left(\frac{I_1I_2}{d}\right) = \left(\frac{c^2}{v^2}\right)\frac{\mu_0}{2\pi}\left(\frac{I_1I_2}{d}\right) = \frac{c^2}{v^2} f_m, \text{ where}$$

$c \equiv 1/\varepsilon_0\mu_0 = 3.00\times10^8\, m/s.$ Here $\frac{f_e}{f_m} = \frac{c^2}{v^2} = \left(\frac{3.0\times10^{10}}{9.1\times10^{-3}}\right)^2 = \boxed{1.1\times10^{25}.}$

$f_e = 1.1\times10^{25} \quad 2\times10^{-7} \quad = \boxed{2\times10^{18}\, N/cm}$

Problem 5.20

Ampere's law says $\nabla\times B = \mu_0 J$. Together with the continuity equation (5.29) this gives $\nabla. \ \nabla\times B \ = \mu_0\nabla.J = -\mu_0\partial\rho/\partial t$, which is inconsistent with unless is constant (magnetostatics) . The other Maxwell equations are OK : $\nabla\times E = 0 \Rightarrow \nabla. \ \nabla\times E \ = 0$, and ac for the two divergence equations , there is no relevant vanishing second derivative (the other one is curl (grad) , which doesn't involve the divergence) .

Problem 5.21

At this stage I'd expect no changes in Gauss's law or Ampere's law . The divergence of B would take the form $\boxed{\nabla\cdot B = \alpha_0\rho_m}$, where ρ_m is the density of magnetic charge , and α_0 is some constant (analogous to and μ_0) . The curl of E becomes $\boxed{\nabla\times E = \beta_0 J_m}$, where J_m is the magnetic current density (representing the flow of magnetic charge) , and β_0 is another constant . Presumably magnetic charge is conserved , so ρ_m and J_m satisfy a continuity equation : $\nabla\cdot J_m = -\partial\rho_m/\partial t$

As for the Lorentz force law , one might guess something of the form $q_m\left[B + \ v\times E \ \right]$ (where q_m is the magnetic charge) . But this is dimensionally impossible , since E has the same units as υB . Evidently we need to divide $v\times E$ by something with the dimensions of velocity – squared . The natural candidate is $c^2 = 1/\varepsilon_0\mu_0$: $\boxed{F = q_e\left[E + \ \nu \times B\ \right] + q_m\left[B - \frac{1}{c^2}\ \nu \times E\ \right]}$ .In this form the magnetic analog to Coulomb's law reads $F = \frac{\alpha_0}{4\pi}\frac{q_{m_1}q_{m_2}}{r^2}\hat{r}$, so to determine we would first introduce (arbitraily) a unit of magnetic charge, then measure the force between unit charges at a given separation. [for further details, and an explanation of the minus sign in the force law, see Prob. 7.35]

Problem 5.22

$$A = \frac{\mu_0}{4\pi}\int \frac{I\hat{z}}{r}dz = \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2}\frac{dz}{\sqrt{z^2+s^2}}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\ \ z + \sqrt{z^2+s^2}\ \ \right]\Bigg|_{z_1}^{z_2} = \boxed{\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2 + \sqrt{\ z_2\ ^2 + s^2}}{z_1 + \sqrt{\ z_1\ ^2 + s^2}}\right]\hat{z}}$$

$$B = \nabla \times A = -\frac{\partial A}{\partial s}\hat{\phi} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2 + \sqrt{\ z_2\ ^2+s^2}}\frac{s}{\sqrt{\ z_2\ ^2+s^2}} - \frac{1}{z_1 + \sqrt{\ z_1\ ^2+s^2}}\frac{s}{\sqrt{\ z_1\ ^2+s^2}}\right]\hat{\phi}$$

$$= -\frac{\mu_0 Is}{4\pi}\left[\frac{z_2 - \sqrt{\ z_2\ ^2+s^2}}{z_2\ ^2 - \ z_2\ ^2 + s^2}\frac{1}{\sqrt{\ z_2\ ^2+s^2}} - \frac{z_2 - \sqrt{\ z_2\ ^2+s^2}}{z_2\ ^2 - \ z_2\ ^2 + s^2}\frac{1}{\sqrt{\ z_2\ ^2+s^2}} - \frac{z_1 - \sqrt{\ z_1\ ^2+s^2}}{z_1\ ^2 - \ z_1\ ^2 + s^2}\frac{1}{\sqrt{\ z_1\ ^2+s^2}}\right]\hat{\phi}$$

$$= -\frac{\mu_0 Is}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{\ z_2\ ^2+s^2}} - 1 - \frac{z_1}{\sqrt{\ z_1\ ^2+s^2}} + 1\right]\hat{\phi} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - \frac{z_1}{\sqrt{z_1^2+s^2}}\right]\hat{\phi},$$

Or , since $\sin\theta_1 = \frac{z_1}{\sqrt{\ z_1\ ^2+s^2}}$ and $\sin\theta_2 = \frac{z_2}{\sqrt{\ z_2\ ^2+s^2}}$,

$\boxed{\frac{\mu_0 I}{4\pi s}\ \sin\theta_2 - \sin\theta_1\ \hat{\phi}}$ (as in Eq. 5.35).

Problem 5.23

$A_\phi = k \Rightarrow B = \nabla \times A = \frac{1}{s}\frac{\partial}{\partial s}\ \ sk\ \ \hat{z}; J = \frac{1}{\mu_0}\ \ \nabla \times B\ \ = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \boxed{\frac{k}{\mu_0 s^2}\hat{\phi}}$.

Problem 5.24

$\nabla \cdot A = -\frac{1}{2}\nabla \cdot\ \ r \times B\ \ = -\frac{1}{2}\left[B \cdot\ \ \nabla \times r\ \ - r \cdot\ \ \nabla \times B\ \ \right] = 0$ since $\nabla \times B = 0$ (B is uniform) and $\nabla \times r = 0$

(Prob. 1.62) . $\nabla \times A = -\frac{1}{2}\nabla \times\ \ r \times B\ \ = -\frac{1}{2}\left[\ \ B \cdot \nabla\ \ r -\ \ r \cdot \nabla\ \ B + r\ \ \nabla \cdot B\ \ - B\ \ \nabla \cdot r\ \ \right]$. But

$r \cdot \nabla\ \ B = 0$ and $\nabla \cdot B = 0$ (since B is uniform) , and $\nabla \cdot r = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$.

Finally ,

$B \cdot \nabla\ \ r = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)\ \ x\hat{x} + y\hat{y} + z\hat{z}\ \ = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} = B$. So

$\nabla \times A = -\frac{1}{2}\ \ B - 3B\ \ = B$.

Problem 5.25

(a) A points is the same direction as I , and is a function anly of s (the distance from the wire) . In cylindrical coordintes , then , $A = A\ \ s\ \ \hat{z}$, so

$B = \nabla \times A = -\frac{\partial A}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ (the field of an infinite wire) .Therefore $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi s}$ and

$\boxed{A\ \ r\ \ = -\frac{\mu_0 I}{2\pi}\ln\ \ s/a\ \ \hat{z}}$ (the constant is arbitrary ; you could use 1 , but then the units look fishy) . $\nabla \cdot A = \frac{\partial A_z}{\partial z} = 0.$ $\nabla \times A = -\frac{\partial A_z}{\partial z}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = B$.

(b) Here Ampere's law gives $\oint B \cdot dl = B 2\pi s = \mu_0 I_{enc} = \mu_0 J \pi s^2 = \mu_0 \frac{I}{\pi R^2}\pi s^2 = \frac{\mu_0 I s^2}{R^2}$.

$B = \frac{\mu_0}{2\pi}\frac{Is}{R^2}\hat{\phi}$. $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi}\frac{s}{R^2} \Rightarrow A = -\frac{\mu_0 I}{4\pi R^2}\ \ s^2 - b^2\ \ \hat{z}$ .Here b is again arbitraty , except that since A must be continuous at R ,

$-\frac{\mu_0 I}{2\pi}\ln\ R/a\ = -\frac{\mu_0 I}{4\pi R^2}\ R^2 - b^2$ which means that we must pick a and b such that

$2\ln\ R/b\ = 1 - \ b/R\ ^2$ . . I'll use a=b=R . Then $\boxed{A = \begin{cases} -\frac{\mu_0 I}{4\pi R^2}\ s^2 - R^2\ \hat{z}, \text{for } s \le R \\ -\frac{\mu_0 I}{2\pi}\ln\ s/R\ \hat{z}, \quad \text{for } s \ge R \end{cases}}$

Problem 5.26

$K = K\hat{x} \Rightarrow B = \pm\frac{\mu_0 K}{2}\hat{y}$ (plus for z < 0 , minus for z > 0) .

A is parallel to K , and depends only on z , so $A = A\ z\ \hat{x}$

$$B = \nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A\ z & 0 & 0 \end{vmatrix} = \frac{\partial A}{\partial z}\hat{y} = \pm\frac{\mu_0 K}{2}\hat{y}$$

$\boxed{A = -\frac{\mu_0 K}{2}|z|\hat{x}}$ will do the job-or this plus any constant .

Problem 5.27

(a) $\nabla \cdot A = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{J}{r}\right) = \frac{1}{r}\ \nabla \cdot J\ + J \cdot \nabla\left(\frac{1}{r}\right)$. But the first term is zero , because $J\ r'$ is a function of the source coordinates , not the field coordinates . And since $r = r - r', \nabla\left(\frac{1}{r}\right) = -\nabla'\left(\frac{1}{r}\right)$. So $\nabla \cdot \left(\frac{J}{r}\right) = -J \cdot \nabla'\left(\frac{1}{r}\right)$. But $\nabla' \cdot \left(\frac{J}{r}\right) = \frac{1}{r}\ \nabla' \cdot J\ + J \cdot \nabla'\left(\frac{1}{r}\right)$, and $\nabla' \cdot J = 0$ in magnetostatics (Eq. 5.31) So $\nabla . \left(\frac{J}{r}\right) = -\nabla' \cdot \left(\frac{J}{r}\right)$, and hence , by the divergence theorem ,

$\nabla \cdot A = -\frac{\mu_0}{4\pi}\int \nabla' \cdot \left(\frac{J}{r}\right) d\tau' = -\frac{\mu_0}{4\pi}\oint \frac{J}{r} \cdot da'$ where the integral is now over the surface surrounding all the currents . But J = 0 on this surface , so $\nabla \cdot A = 0$

(b) $\nabla\times A=\frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{J}{r}\right)d\tau'=\frac{\mu_0}{4\pi}\int\left[\frac{1}{r}\ \nabla\times J\ \ -J\times\nabla\left(\frac{1}{r}\right)d\tau'\right]$. But $\nabla\times J=0$ (since J is not a function of r) and $\nabla\left(\frac{1}{r}\right)=-\frac{\hat{r}}{r^2}$ (Eq. 1.101) , so $\nabla\times A=\frac{\mu_0}{4\pi}\int\frac{J\times\hat{r}}{r^2}d\tau'=B$

(c) $\nabla^2 A=\frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{J}{r}\right)dr'$. But $\nabla^2\left(\frac{J}{r}\right)=J\nabla^2\left(\frac{1}{r}\right)$ (once again , J is a constant , as far as differenti-ation with respect to r is concerned , and $\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta^3\ \ r$ (Eq . 1.102)

So $\nabla^2 A=\frac{\mu_0}{4\pi}\int J\ \ r'\ \left[-4\pi\delta^3\ \ r\ \ \right]d\tau'=-\mu_0 J\ \ r$

Problem 5.28

$\mu_0 I=\oint B\cdot dl=-\int_a^b\nabla U\cdot dl=-\left[U\ \ b\ \ -U\ \ a\ \ \right]$(by the gradient theorem) , so

$U\ \ b\ \#U\ \ a$. qed

For an infinite straight wire , $B=\frac{\mu_0 I}{2\pi s}\hat{\phi}$. $\boxed{U=-\frac{\mu_0 I\phi}{2\pi}}$, would do the job , in the sense that $-\nabla U=\frac{\mu_0 I}{2\pi}\nabla\ \ \phi\ \ =\frac{\mu_0 I}{2\pi}\frac{1}{s}\frac{\partial\phi}{\partial\phi}\hat{\phi}=B$. But when advances by , this function does not return to its initial value ; it works (say) for $0\le\phi<2\pi$, but at 2π it "jumps" back to zero.

Problem 5.29

Use Eq. 5.67 , with $R\to\bar{r}$ and $\sigma\to\rho d\bar{r}$:

$$A=\frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\phi}\int_0^r\bar{r}d\bar{r}+\frac{\mu_0\omega\rho}{3}r\sin\theta\hat{\phi}\int_r^R\bar{r}d\bar{r}$$

$$=\left(\frac{\mu_0\omega\rho}{3}\right)\sin\theta\left[\frac{1}{r^2}\left(\frac{r^5}{5}\right)+\frac{r}{2}\ \ R^2-r^2\ \ \right]\hat{\phi}=\frac{\mu_0\omega\rho}{2}r\sin\theta\left(\frac{R^2}{3}-\frac{r^2}{5}\right)\hat{\phi}$$

$$B=\nabla\times A=\frac{\mu_0\omega\rho}{2}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta r\sin\theta\left(\frac{R^2}{3}-\frac{r^2}{5}\right)\right]\hat{r}-\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\sin\theta\left(\frac{R^2}{3}-\frac{r^2}{5}\right)\right]\hat{\theta}\right\}$$

$$=\mu_0\omega\rho\left[\left(\frac{R^2}{3}-\frac{r^2}{5}\right)\cos\theta\hat{r}-\left(\frac{R^2}{3}-\frac{2r^2}{5}\right)\sin\theta\hat{\theta}\right]$$ But $\rho=\frac{Q}{(4/3)\pi R^3}$, so

$$=\boxed{\frac{\mu_0\omega Q}{4\pi R}\left[\left(1-\frac{3r^2}{5R^2}\right)\cos\theta\hat{r}-\left(1-\frac{6r^2}{5R^2}\right)\sin\theta\hat{\theta}\right]}$$

Problem 5.30

(a)

$$\left\{\begin{array}{l} -\dfrac{\partial W_z}{\partial x}=F_y \Rightarrow W_z(x,y,z)=-\displaystyle\int_0^x F_y(x',y,z)\,dx'+C_1(y,z). \\ \dfrac{\partial W_y}{\partial x}=F_z \Rightarrow W_y(x,y,z)=+\displaystyle\int_0^x F_z(x',y,z)\,dx'+C_2(y,z). \end{array}\right\}$$

These satisfy (ii) and (iii) , for any C_1 and C_2 ; it remains to choose these functions so as to satisfy (i) :

$$-\int_0^z \frac{\partial F_y(x',y,z)}{\partial y}dx'+\frac{\partial C_1}{\partial y}-\int_0^z \frac{\partial F_z(x',y,z)}{\partial z}dx'-\frac{\partial C_2}{\partial z}=F_x(x,y,z)$$ But $\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}=0$, so

$$\int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}dx'+\frac{\partial C_1}{\partial y}-\frac{\partial C_2}{\partial z}=F_x(x,y,z)$$. Now $\int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}dx'=F_x(x,y,z)-F_x(0,y,z)$

, so $\frac{\partial C_1}{\partial y}-\frac{\partial C_2}{\partial z}=F_x(0,y,z)$. We may as well pick $C_2=0$, $C_1(y,z)=\int_0^y F_x(0,y',z)\,dy'$

and we're done , with $W_z=0$; $W_y=\int_0^y F_z(x',y,z)\,dx'-\int_0^x F_y(x',y,z)\,dx'$;

$$W_z=\int_0^y F_x(0,y',z)-\int_0^x F_y(x',y,z)\,dx'$$

(b) $\nabla\times W=\left(\frac{\partial W_z}{\partial y}-\frac{\partial W_y}{\partial z}\right)\hat{x}+\left(\frac{\partial W_x}{\partial z}-\frac{\partial W_z}{\partial x}\right)\hat{y}+\left(\frac{\partial W_y}{\partial x}-\frac{\partial W_x}{\partial y}\right)\hat{z}$

$$=\left[F_x(0,y,z)-\int_0^x \frac{\partial F_y(x',y,z)}{\partial y}dx'-\int_0^x \frac{\partial F_z(x',y,z)}{\partial z}dx'\right]\hat{x}+\left[0+F_y(x,y,z)\right]\hat{y}+\left[F_z(x,y,z)-0\right]\hat{z}$$

But , $\nabla \cdot F = 0$ so the $\hat{x}$ term is

$$\left[F_x \; 0,y,z \; + \int_0^x \frac{\partial F_x \;\; x',y,z}{\partial x'} dx' \right] = F_x \; 0,y,z \; + F_x \; x,y,z \; - F_x \; 0,y,z$$

So $\nabla \times W = F$.

$$\nabla \cdot W = \frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z} = 0 + \int_0^x \frac{\partial F_z \;\; x',y,z}{\partial y} + \int_0^y \frac{\partial F_x \;\; 0,y',z}{\partial z} - \int_0^x \frac{\partial F_y \;\; x',y,z}{\partial z} dx' \# 0$$

(c) $W_y = \int_0^x x' dx' = \frac{x^2}{2}; W_z = \int_0^y y' dy' - \int_0^y z dx' = \frac{y^2}{2} - zx$.

$$\boxed{W = \frac{x^2}{2} \hat{y} + \left(\frac{y^2}{2} - zx \right) \hat{z}} . \quad \nabla \times W = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^2/2 & y^2/2 - zx \end{vmatrix} = y\hat{x} + z\hat{y} + x\hat{z} = F$$

Problem 5.31

(a) At the surface of the solenoid , $B_{above} = 0, B_{below} = \mu_0 nI\hat{z} = \mu_0 K\hat{z}; \hat{n} = \hat{s};$; so $K \times \hat{n} = -K\hat{z}$ Evidently Eq. 5.74 holds.

(b) In oEq. 5.67 , both expressions reduce to $\mu_0 R^2 \omega\sigma/3 \; \sin\theta\hat{\phi}$ at the surface , so Eq. 5.75 is satisfied .

$\left.\frac{\partial A}{\partial r}\right|_{R^+} = \frac{\mu_0 R^4 \omega\sigma}{3} \left(-\frac{2\sin\theta}{r^3} \right) \hat{\phi} \Big|_R = -\frac{2\mu_0 R\omega\sigma}{3} \sin\theta\hat{\phi}$; $\left.\frac{\partial A}{\partial r}\right|_{R^-} = \frac{\mu_0 R\omega\sigma}{3} \sin\theta\hat{\phi}$. So the left side of Eq. 5.76 is $-\mu_0 R\omega\sigma \sin\theta\hat{\phi}$. Meanwhile $K = \sigma v = \sigma \;\; \omega \times r \;\; = \sigma\omega R \sin\theta\hat{\phi}$, so the right side of Eq. 5.76 is $-\mu_0 \sigma\omega R \sin\theta\hat{\phi}$, and the equation is satisfied .

Problem 5.32

Because $A_{above} = A_{below}$ at every point on the surface , it follows that $\frac{\partial A}{\partial x}$ and $\frac{\partial A}{\partial y}$ are the same above and below , any discontinuity is confined to the norm derivative .

$B_{above} - B_{below} = \left(-\frac{\partial A_{y_{above}}}{\partial z} + \frac{\partial A_{y_{below}}}{\partial z} \right)\hat{x} + \left(\frac{\partial A_{x_{above}}}{\partial z} - \frac{\partial A_{x_{below}}}{\partial z} \right)\hat{y}$. But Eq. 5.74 says this equals $\mu_0 K(-\hat{y})$. So $\frac{\partial A_{x_{above}}}{\partial z} - \frac{\partial A_{x_{below}}}{\partial z} = -\mu_0 K$. Thus the normal derivativeof the com - ponent of A parallel to K surfers a discontinuity $-\mu_0 K$, or , more compactly :

$$\boxed{\frac{\partial A_{above}}{\partial n} - \frac{\partial A_{below}}{\partial n} = -\mu_0 K}$$

Problem 5.33

(Same idea as Prob. 3.33.) Write $m = (m\cdot\hat{r})\hat{r} + (m\cdot\hat{\theta})\hat{\theta} = mcos\theta\hat{r} - m\sin\theta\hat{\theta}$ (Fig. 5.54) . Then $3(m\cdot\hat{r})\hat{r} - m = 3m\cos\theta\hat{r} - m\cos\theta\hat{r} + m\sin\theta\hat{\theta} = 2m\cos\theta\hat{r} + m\sin\theta\hat{\theta}$, and Eq. 5.87

Problem 5.34

(a) $m = Ia = \boxed{I\pi R^2\hat{z}}$.

(b) $B \approx \boxed{\frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})}$

(c) On the z axis , $\theta = 0, r = z, \hat{r} = \hat{z}$ (for z > 0) $\boxed{B \approx \frac{\mu_0 IR^2}{2z^3}\hat{z}}$ (for $z < 0, \theta = \pi, \hat{r} = -\hat{z}$, so the field is the same , with $|z|^3$ in place of z^3) . The exact answer (Eq. 5.38) reduces (for $z \gg R$) to $B \approx \mu_0 IR^2/2|z|^3$ so they agree .

Problem 5.35

For a ring , $m = I\pi r^2$ Here $I \to \sigma\upsilon dr = \sigma\omega r dr$,so $m = \int_0^R \pi r^2\sigma\omega r dr = \boxed{\pi\omega\sigma R^4/4}$

Problem 5.36

The total charge on the shaded ring is $dq = \sigma(2\pi R\sin\theta)Rd\theta$

The time for one revolution is $dt = 2\pi/\omega$. So the current in the ring is $I = \frac{dq}{dt} = \sigma\omega R^2\sin\theta d\theta$. The area of the ring is $\pi(R\sin\theta)^2$, so the magnetic moment of the ring is $dm = \sigma\omega R^2 \sin\theta d\theta \; \pi R^2 \sin^2\theta$, and the total dipole moment of the shell is $m = \sigma\omega R^4 \int_0^\pi \sin^3\theta d\theta = 4/3 \; \sigma\omega\pi R^4$, or $\boxed{m = \frac{4\pi}{3}\sigma\omega R^4 \hat{z}}$

The dipole term in the multipole expansion for A is there – fore

$A_{dip} = \frac{\mu_0}{4\pi}\frac{4\pi}{3}\sigma\omega R^4 \frac{\sin\theta}{r^2}\hat{\phi} = \frac{\mu_0\sigma\omega R^4}{3}\frac{\sin\theta}{r^2}\hat{\phi}$, which is also the exact potential (Eq. 5.67) ; evidently a spinning sphere produces a perfect dipole field , with no higher multipole con – tributions .

Problem 5.37

The field of one side is given by Eq. 5.35 , with $s \rightarrow \sqrt{z^2 + w/2^{\;2}}$ and

$\sin\theta_2 = -\sin\theta_1 = \frac{w/2}{\sqrt{z^2 + w^2/2}}$; $B = \frac{\mu_0 I}{4\pi}\frac{w}{\sqrt{z^2 + w^2/4}\sqrt{z^2 + w^2/2}}$. To pick off the vertical compoment , multiply by $\sin\phi = \frac{w/2}{\sqrt{z^2 + w^2/2}}$; for all four sides , multiply by 4 : $\boxed{B = \frac{\mu_0 I}{2\pi}\frac{w^2}{\sqrt{z^2 + w^2/4}\sqrt{z^2 + w^2/2}}\hat{z}}$. For $z \gg w, B \approx \frac{\mu_0 I w^2}{2\pi z^3}\hat{z}$. The field of a dipole $\boxed{m = Iw^2}$, for points on the z axis (Eq. 5.86 , with $r \rightarrow z, \hat{r} \rightarrow \hat{z}, \theta = 0$) is $B = \frac{\mu_0}{2m}\frac{m}{z^3}\hat{z}$

Problem 5.38

The mobile charges do pull in toward the axis , but the resulting concentration of (negative) charge sets up an electric field that repels away further accumulation . Equilibrium is reached when the electric repulsion on a mobile charge q balances the magnetic attraction : $F = q\left[E + \nu \times B\right] = 0 \Rightarrow E = -\nu \times B$. Say the current is in the z direction : $J = \rho - \upsilon\hat{z}$ (where $\rho\_$ and υ are both negative)

$\oint B \cdot dl = B 2\pi s = \mu_0 J \pi s^2 \Rightarrow B = \frac{\mu_0 \rho - \upsilon s}{2}\hat{\phi}$

$$\int E\cdot da = E2\pi sl = \frac{1}{\varepsilon_0}\left(\rho_+ + \rho_-\right)\pi s^2 l \Rightarrow E = \frac{1}{2\varepsilon_0}\left(\rho_+ + \rho_-\right)s\hat{s}$$

$$\frac{1}{2\varepsilon_0}\left(\rho_+ + \rho_-\right)s\hat{s} = -\left[\left(\upsilon\hat{z}\right)\times\left(\frac{\mu_0\rho_-\upsilon s}{2}\hat{\phi}\right)\right] = \frac{\mu_0}{2}\rho_-\upsilon^2 s\hat{s} \Rightarrow \rho_+ + \rho_- = \rho_-\left(\varepsilon_0\mu_0\upsilon^2\right) = \rho_-\left(\frac{\upsilon^2}{c^2}\right)$$

Evidently $\rho_+ = -\rho_-\left(1-\frac{\upsilon^2}{c^2}\right) = \frac{\rho_-}{\gamma^2}$, or $\rho_- = -\gamma^2\rho_+$. In this naïve model , the mobile negative charges fill a smaller inner cylinder , leaving a shell of positive (stationary) charge at the outside . But since $\upsilon \ll c$, the effect is extremely small .

Problem 5.39

(a) If positive charges flow to the right , they are deflected $\boxed{\text{down}}$, and the bottom plate acquires a positive charge .

(b) $q\upsilon B = qE \Rightarrow E = \upsilon B \Rightarrow V = Et = \boxed{\upsilon Bt}$ with the bottom at higher potential .

(c) If negative charges flow to the left , they are also deflected down , and the bottom plate acquires a negative charge . The potential difference is still the same , but this time the top plate is at the higher potential .

Problem 5.40

From Eq. 5.17 , $F = I\int\left(dl\times B\right)$. But B is constant , in this case , so it comes outside the integral : $F = I\left(\int dl\right)\times B$ and $\int dl =$w , the vecto displacement from the point at which the wire first enters the field to the point where it leaves . Since w and B are perpendicular , $F = IB\omega$ and F is perpendicular to w .

Problem 5.41

The angular momentum acquired by the particle as it moves out from the center to the edge is

$$L = \int\frac{dL}{dt}dt = \int N dt = \int\left(r\times F\right)dt = \int r\times q\left(v\times B\right)dt = q\int r\times\left(dl\times B\right) = q\left[\int\left(r\cdot B\right)dl - \int B\left(r\cdot dl\right)\right]$$

But r is perpendicular to B , so $r\cdot B = 0$, and

$$r\cdot dl = r\cdot dr = \frac{1}{2}d\left(r\cdot r\right) = \frac{1}{2}d\left(r^2\right) = rdr = \left(1/2\pi\right)\left(2\pi rdr\right)$$

So $L=-\frac{q}{2\pi}\int_0^R B2\pi r dr=-\frac{q}{2\pi}\int Bda$. It follows that $\boxed{L=-\frac{q}{2\pi}\phi}$, where $\phi=\int Bda$ is the total flux . In particular , if $\phi=0$, then $L=0$, and the charge emerges with zero angular momentum , which means it is going along a radial line . qed

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Problem 5.42

From Eq. 5.42 , $F=\int K\times B_{ave}\ da$. Here $K=\sigma\nu,\nu=\omega R\sin\theta d\theta d\phi$, and $B_{ave}=\frac{1}{2}\ B_{in}+B_{out}$. From Eq. 5.68 , $B_{in}=\frac{2}{3}\mu_0\sigma R\omega\hat{z}=\frac{2}{3}\mu_0\sigma R\omega\ \cos\theta\hat{r}-\sin\theta\hat{\theta}$. From Eq. 5.67 ,

$$B_{out}=\nabla\times A=\nabla\times\left(\frac{\mu_0R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi}\right)=\frac{\mu_0R^4\omega\sigma}{3}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\hat{\theta}\right],$$

$$=\frac{\mu_0R^4\omega\sigma}{3r^3}\ 2\cos\theta\hat{r}+\sin\theta\hat{\theta}\ =\frac{\mu_0R\omega\sigma}{3}\ 2\cos\theta\hat{r}+\sin\theta\hat{\theta}\quad(\text{ since } r=R\ )$$

$$B_{ave}=\frac{\mu_0R\omega\sigma}{6}\ 4\cos\theta\hat{r}-\sin\theta\hat{\theta}$$

$$K\times B_{ave}=\ \sigma\omega R\sin\theta\ \left(\frac{\mu_0R\omega\sigma}{6}\right)\left[\hat{\phi}\times\ 4\cos\theta\hat{r}-\sin\theta\hat{\theta}\ \right]=\frac{\mu_0}{6}\ \sigma\omega R\ ^2\ 4\cos\theta\hat{\theta}+\sin\theta\hat{r}\ \sin\theta$$

Picking out the z compoment of (namely ,) and of (namely ,) , we have

$$K\times B_{ave}\ _z=-\frac{\mu_0}{2}\ \sigma\omega R\ ^2\sin^2\theta\cos\theta$$

$$F_z=-\frac{\mu_0}{2}\ \sigma\omega R\ ^2R^2\int\sin^3\theta\cos\theta d\theta d\phi=-\frac{\mu_0}{2}\ \sigma\omega R^2\ ^2 2\pi\left(\frac{\sin^4\theta}{4}\right)\Bigg|_0^{\pi/2}\text{ , or}$$

$$\boxed{F=-\frac{\mu_0\pi}{4}\ \sigma\omega R^2\ ^2\hat{z}}$$

Problem 5.43

(a) $F=ma=q_e\ \nu\times B\ =\frac{\mu_0}{4\pi}\frac{q_eq_m}{r^2}\ \nu\times\hat{r}$; $\boxed{a=\frac{\mu_0}{4\pi}\frac{q_eq_m}{mr^3}\ \nu\times r}$

(b) Because . But $a\perp\nu,a\cdot\nu=\frac{1}{2}\frac{d}{dt}\ \nu\cdot\nu\ =\frac{1}{2}\frac{d}{dt}\ \upsilon^2\ =\upsilon\frac{d\upsilon}{dt}$. So $\frac{d\upsilon}{dt}=0$. qed

(c)

$$\frac{dQ}{dt}=m\ \ \nu\times\nu\ \ +m\ \ r\times a\ \ -\frac{\mu_0 q_e q_m}{4\pi}\frac{d}{dt}\left(\frac{r}{r}\right)=0+\frac{\mu_0 q_e q_m}{4\pi r^3}\left[r\times\ \ \nu\times r\ \ \right]-\frac{\mu_0 q_e q_m}{4\pi}\left(\frac{\nu}{r}-\frac{r}{r^2}\frac{dr}{dt}\right)$$ [150]

$$=\frac{\mu_0 q_e q_m}{4\pi}\left\{\frac{1}{r^3}\left[r^2\nu-\ \ r\cdot\nu\ \ r\right]-\frac{\nu}{r}+\frac{r}{r^2}\frac{d}{dt}\ \ \sqrt{r\cdot r}\ \ \right\}=\frac{\mu_0 q_e q_m}{4\pi}\left[\frac{\nu}{r}-\frac{\hat{r}\cdot\nu}{r}\hat{r}-\frac{\nu}{r}+\frac{\hat{r}}{2r}\frac{2\ \ r\cdot\nu}{r}\right]=0$$

(d) (i) $Q\cdot\hat{\phi}=Q\ \ \hat{z}\cdot\hat{\phi}\ \ =m\ \ r\times\nu\ \ \cdot\hat{\phi}-\frac{\mu_0 q_e q_m}{4\pi}\ \ \hat{r}\cdot\hat{\phi}$. But $\hat{z}\cdot\hat{\phi}=\hat{r}\cdot\hat{\phi}=0$, so $r\times\nu\ \ \cdot\hat{\phi}=0$. But $r=r\hat{r}$, and $\nu=\frac{dl}{dt}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}+r\sin\theta\dot{\phi}\hat{\phi}$ (where dots denote differentiation with respect to time) , so

$$r\times\nu=\begin{vmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\\ r & 0 & 0\\ \dot{r} & r\dot{\theta} & r\sin\theta\dot{\phi}\end{vmatrix}=\ -r^2\sin\theta\dot{\phi}\ \ \hat{\theta}+\ \ r^2\dot{\theta}\ \ \hat{\phi}$$

Therefore $r\times\nu\ \ \cdot\hat{\phi}=r^2\dot{\theta}=0$, so is constant . qed

(ii) $Q.\hat{r}=Q\ \ \hat{z}\cdot\hat{r}\ \ =m\ \ r\times\nu\ \ \cdot\hat{r}-\frac{\mu_0 q_e q_m}{4\pi}\ \ \hat{r}\cdot\hat{r}$ But $\hat{z}\cdot\hat{r}=\cos\theta$, and $r\times\nu\ \ \perp r\Rightarrow\ \ r\times\nu\ \ \cdot\hat{r}=0$, so $Q\cos\theta=-\frac{\mu_0 q_e q_m}{4\pi}$, or $Q=-\frac{\mu_0 q_e q_m}{4\pi\cos\theta}$. And since θ constant , so too is Q . qed

(iii) $Q\cdot\hat{\theta}=Q\ \ \hat{z}\cdot\hat{\theta}\ \ =m\ \ r\cdot\nu\ \ \hat{\theta}-\frac{\mu_0 q_e q_m}{4\pi}\ \ \hat{r}\cdot\hat{\theta}$. But $\hat{z}\cdot\hat{\theta}=-\sin\theta,\hat{r}\cdot\hat{\theta}=0$, and $r\times\nu\ \ \cdot\hat{\theta}=-r^2\sin\theta\dot{\phi}$ (from (i)) , so $-Q\sin\theta=-mr^2\sin\theta\dot{\phi}\Rightarrow\dot{\phi}=\frac{Q}{mr^2}=\frac{k}{r^2}$, with

$$\boxed{k=\frac{Q}{m}=-\frac{\mu_0 q_e q_m}{4\pi\cos\theta}}$$

(e) $\upsilon^2=\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2$, but $\dot{\theta}=0$ and $\dot{\phi}=\frac{k}{r^2}$, so

$$\dot{r}^2=\upsilon^2-r^2\sin^2\theta\frac{k^2}{r^4}=\upsilon^2-\frac{k^2\sin^2\theta}{r^2}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{\upsilon^2 - (k\sin\theta/r)^2}{k^2/r^4} = r^2\left[\left(\frac{\upsilon r}{k}\right)^2 - \sin\theta\right]; \boxed{\frac{dr}{d\phi} = r\sqrt{\left(\frac{\upsilon r}{k}\right)^2 - \sin^2\theta}}$$

(f) , $\displaystyle\int \frac{dr}{r\sqrt{(\upsilon r/k)^2 - \sin^2\theta}} = \int d\phi \Rightarrow \phi - \phi_0 = \frac{1}{\sin\theta}\sec^{-1}; \sec[(\phi-\phi_0)\sin\theta] = \frac{\upsilon r}{k\sin\theta}$

or $\boxed{r(\phi) = \dfrac{A}{\cos[(\phi-\phi_0)\sin\theta]}}$ where $A = -\dfrac{\mu_0 q_e q_m \tan\theta}{4\pi m\upsilon}$

Problem 5.44

Put the field point on the x axis , so . Then . The x and y compoments integrate to zero (z integrand is odd , as in Prob . 5.17) .

Inside the solenoid , . Outside the solenoid , s > R , so

Here , so (inside) , and 0 (outside) (as we found more easily using Ampere's law , in Ex. 5.9) .

Problem 5.45

Let the source point be , and the field point be ; then and

Problem 5.46

(a) From Eq. 5.38 ,

(b) Differentiating again :

. Zero if , in which case

Problem 5.47

(a) The total charge on the shaded ring is . The time for one revolution is . So the current in the ring is . From Eq. 5.38 , the magnetic field of this ring (for point on the axis) is , and the total field of the disk is

(b) Slice the sphere into slabs of thickness t , and use (a) . Here

.First rewrite the term in square brackets :

But . So . But

Problem 5.48

$B = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \frac{dl' \times r}{r^2}$. $r = -R\cos\phi\hat{x} + (y - R\sin\phi)\hat{y} + z\hat{z}$ (For simplicity I'll drop the prime on ϕ) $r^2 = R^2\cos^2\phi + y^2 - 2Ry\sin\phi + R^2\sin^2\phi + z^2 = R^2 + y^2 + z^2 - 2Ry\sin\phi$ .The source coordinates (x' , y' , z') satisfy

$x' = R\cos\phi \Rightarrow dx' = -R\sin\phi d\phi; y' = R\sin\phi \Rightarrow dy' = R\cos\phi d\phi; z' = 0 \Rightarrow dz' = 0$. So

$dl' = -R\sin\phi d\phi \hat{x} + R\cos\phi d\phi \hat{y}$

$$dl' \times r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\sin\phi d\phi & R\cos\phi d\phi & 0 \\ -R\cos\phi & y - R\sin\phi & z \end{vmatrix} = (Rz\cos\phi\phi)\ \hat{x} + (Rz\sin\phi d\phi)\ \hat{y} + (-Ry\sin\phi d\phi + R^2 d\phi)\ \hat{z}$$

$$B_z = \frac{\mu_0 IRz}{4\pi}\int_0^{2\pi} \frac{\cos\phi d\phi}{(R^2 + y^2 + z^2 - 2Ry\sin\phi)^{3/2}} = \frac{\mu_0 IR}{4\pi}\frac{1}{Ry}\frac{1}{\sqrt{R^2 + y^2 + z^2 - 2Ry\sin\phi}}\Bigg|_0^{2\pi} = 0$$

Since $\phi = 0$ at both limits . The y and z components are elliptic integrals , and cannot be expressed in terms of elementary functions .

$$\boxed{B_z = 0; B_y = \frac{\mu_0 IRz}{4\pi}\int_0^{2\pi} \frac{\sin\phi d\phi}{(R^2 + y^2 + z^2 - 2Ry\sin\phi)^{3/2}}; B_z = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi} \frac{(R - y\sin\phi)\ d\phi}{(R^2 + y^2 + z^2 - 2Ry\sin\phi)^{3/2}}}$$

Problem 5.49

From the Biot-Savart law , the field of loop # 1 is $B = \frac{\mu_0 I_1}{4\pi}\oint_1 \frac{dl_1 \times \hat{r}}{r^2}$; the force on loop # 2 is $F = I_2 \oint_2 dl_2 \times B = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{dl_2 \times (dl_1 \times \hat{r})}{r^2}$

. Now $dl_2 \times (dl_1 \times \hat{r}) = dl_1 (dl_2 \cdot \hat{r}) - \hat{r} (dl_1 \cdot dl_2)$, so

$$F = -\frac{\mu_0}{4\pi} I_1 I_2 \left\{ \oint\oint \frac{\hat{r}}{r^2} (dL_1 . dL_2) - \oint dL_1 \oint \frac{(dL_2 . \hat{r})}{r^2} \right\}$$

The first term is what we want. It remains to show that the second term is zero:

$\mathbf{r} = (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z}$, so

$$\nabla_2 (1/r) = \frac{\partial}{\partial x_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\hat{x} + \frac{\partial}{\partial y_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\hat{y} +$$

$$\frac{\partial}{\partial z_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\hat{z}$$

$$= -\frac{(x_2 - x_1)}{r^3}\hat{x} - \frac{(y_2 - y_1)}{r^3}\hat{y} - \frac{(z_2 - z_1)}{r^3}\hat{z}$$

$$= -\frac{\mathbf{r}}{r^3} = -\frac{\hat{r}}{r^2}$$

So $\oint \frac{\hat{r}}{r^2}\, dL_2 = 0$ (by corollary 2 in sect. 1.3.3). qed

Problem 5.50

Poisson's equation (Eq. 2.24) Says $\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$. For dielectrics (with no free charge), $\rho_b = -\nabla . P$ (Eq.4.12), end the resulting potential is $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{P(\mathbf{r}').\hat{r}}{r^2}\, dt'$. In general, $\rho = \varepsilon_0 \nabla$. E (gauss's law), so the analogy is P → $-\varepsilon_0 E$, and hence $V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{E(\mathbf{r}').\hat{r}}{r^2}\, dt'$.
qed
[there are many other ways to obtain this result. For example, using Eq. 1.100:

$$\nabla . \left(\frac{\hat{r}}{r^2}\right) = -\nabla' . \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\mathbf{r}) = 4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

$$V(\mathbf{r}) = \int V(\mathbf{r}')\delta^3(\mathbf{r} - \mathbf{r}')dt' = -\frac{1}{4\pi}\int V(r')\nabla' . (\frac{\hat{r}}{r^2})dt' = \frac{1}{4\pi}\int \frac{\hat{r}}{r^2} . [\nabla' V(\mathbf{r}')]dt' - \frac{1}{4\pi}\oint V(\mathbf{r}')\frac{\hat{r}}{r^2} . da'$$

(Eq. 1.59). but $\nabla' V(\mathbf{r}') = -E(\mathbf{r}')$, end the surface integral → 0 at ∞, so $V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{E(r').\hat{r}}{r^2} dt'$, as before. You can also check the result, by computing its gradient – but it's not easy.]

Problem 5.51

(a) For uniform B, $\int_0^r (B \times dl) = B \times \int_0^r dl = B \times r \neq A = -\frac{1}{2}(B \times r)$

(b) $B = \frac{\mu_0 I}{2\pi s}\hat{\phi}$, so

$$\oint B \times dl = \left(\frac{\mu_0 I}{2\pi a}\hat{s} - \frac{\mu_0 I}{2\pi b}\hat{s}\right)\omega = \frac{\mu_0 I \omega}{2\pi}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{s} \neq 0$$

(c) $A = -r \times B \int_0^1 \lambda d\lambda = -\frac{1}{2}(r \times B)$

(d) $B = \frac{\mu_0 I}{2\pi s}\hat{\phi}; B(\lambda r) = \frac{\mu_0 I}{2\pi \lambda s}\hat{\phi}; A = -\frac{\mu_0 I}{2\pi s}(r \times \hat{\phi})\int_0^1 \lambda \frac{1}{\lambda} d\lambda = -\frac{\mu_0 I}{2\pi s}(r \times \hat{\phi})$

. But r here is the vector from the origin—in cylindrical coordinates $r = s\hat{s} + z\hat{z}$. So $A = -\frac{\mu_0 I}{2\pi s}\left[s(\hat{s} \times \hat{\phi}) + z(\hat{z} \times \hat{\phi})\right]$, and $(\hat{s} \times \hat{\phi}) = \hat{z}$, $(\hat{z} \times \hat{\phi}) = -\hat{s}$.. So $A = \frac{\mu_0 I}{2\pi s}(z\hat{s} - s\hat{z})$.

The examples in (c) and (d) happen to be divergenceless, but this is not the case in general. For (letting $L \equiv \int_0^1 \lambda B(\lambda r) d\lambda$, for short)

$\nabla.A = -\nabla.(r \times L) = -\left[L.(\nabla \times r) - r.(\nabla \times L)\right] = r.(\nabla \times L)$, and

$\nabla \times L = \int_0^1 \lambda \left[\nabla \times B(\lambda r)\right] d\lambda = \int_0^1 \lambda^2 \left[\nabla_\lambda \times B(\lambda r)\right] d\lambda = \mu_0 \int_0^1 \lambda^2 J(\lambda r) d\lambda$, so

$\nabla.A = \mu_0 r.\int_0^1 \lambda^2 J(\lambda r) d\lambda$, and it vanishes in regions where J = 0 (which is we the examples in (c) and (d) were divergenceless). To construct an explicit counterexample, we need the field at a point where $J \neq 0$—say, inside a wire with uniform current. 105

Here ampére's law gives $B2\pi$ s= $\mu_0 I_{enc} = \mu_0 I\pi s^2 \Rightarrow B = \frac{\mu_0 J}{2} s\hat{\phi}, so$

$$A = -r \times \int_0^1 \lambda \left(\frac{\mu_0 J}{2} \right) \lambda s\hat{\phi} d\lambda = -\frac{\mu_0 I}{6} s\left(r \times \hat{\phi}\right) = \frac{\mu_0 Js}{6}\left(z\hat{s} - s\hat{z}\right)$$

$$\nabla.A = \frac{\mu_0 J}{6}\left[\frac{1}{s}\frac{\partial}{\partial s}\left(s^{2z}\right) - \frac{\partial}{\partial z}\left(s^2\right)\right] = \frac{\mu_0 J}{6}\left(\frac{1}{s}2sz\right) = \frac{\mu_0 Jz}{3} \neq 0$$

Conclusion: (ii) does not automatically yield $\nabla.A = 0.$

Problem 5.52

(a) Exploit the analogy with the electrical case :

$$E = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3\left(P.\hat{r}\right)\hat{r} - P\right] \text{ (eq . 3.104)} = -\nabla V \text{ , with } V = \frac{1}{4\pi\varepsilon_0}\frac{P.\hat{r}}{r^2} \text{ (eq .3.102).}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(m.\hat{r}\right)\hat{r} - m\right] \text{ (eq.5.5.87)} = -\nabla U \text{ , (eq.5.56).}$$

Evidently the prescription is $P/\varepsilon_0 \rightarrow \mu_0 m : U\left(r\right) = \frac{\mu_0}{4\pi}\frac{m.\hat{r}}{r^2}.$

(b) comparing Eqs .5.67 and 5.85 . the dipole moment of the shell is $m = \left(4\pi/3\right)\omega\sigma R^4\hat{z}$ (which we also got in prob .5.36). using the result of (a) , then $, U\left(r\right) = \frac{\mu_0 \omega\sigma R^4}{3}\frac{\cos\theta}{r^2}$ for r>R.

Inside the shell, the field is uniform (eq.5.38): $B = \frac{2}{3}\mu_0 \sigma\omega R\hat{z}$, so

$U(r) = -\frac{2}{3}\mu_0\sigma\omega Rz$ +constant . we may as well pick the constant to be zero , so

$U(r) = -\frac{2}{3}\mu_0\sigma\omega Rr \cos\theta$ for r < R.

[notice that U(r) is not continuous at the surface (r = R) : $U_{in}(R) - -\frac{2}{3}\mu_0\sigma\omega R^2 \cos\theta \neq U_{out}(R) = \frac{1}{3}\mu_0\sigma\omega R^2 \cos\theta$. As I warned you on p.236 : if you insist on using magnetic scalar potentials , keep awap from places where there is current !]

(c)

$$B = \frac{\mu_0\omega Q}{4\pi R}\left[\left(1-\frac{3r^2}{5R^2}\right)\cos\theta\hat{r} - \left(1-\frac{6r^2}{5R^2}\right)\sin\theta\hat{\phi}\right] = -\nabla U = -\frac{\partial U}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial U}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}\hat{\phi}.$$

$$\frac{\partial U}{\partial\phi} = 0 \Rightarrow U(r,\theta,\phi) = U(r,\theta).$$

$$\frac{1}{r}\frac{\partial U}{\partial\theta} = \left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{6r^2}{5R^2}\right)\sin\theta \Rightarrow U(r,\theta) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{6r^2}{5R^2}\right)r\cos\theta + f(r).$$

$$\frac{\partial U}{\partial r} = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{3r^2}{5R^2}\right)\cos\theta \Rightarrow U(r,\theta) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(r-\frac{r^3}{5R^2}\right)\cos\theta + g(\theta).$$

Equating the two expressions :

$$-\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{6r^2}{5R^2}\right)r\cos\theta + f(r) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1-\frac{r^2}{5R^2}\right)r\cos\theta + g(\theta),$$

$$\left(\frac{\mu_0 \omega Q}{4\pi R^3}\right) r^3 \cos\theta + f(r) = g(\theta)$$

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But there is no way to write $r^3 \cos\theta$ as the sum of a function of θ and a function of r, so we're stuck. The reason is that you can't have a scalar magnetic potential in a region where the current is nonzero.

Problem 5.53

(a) $\nabla . B = 0$, $\nabla \times B = \mu_0 J$, and $\nabla . A = 0$,

$\nabla \times A = B \Rightarrow A = \frac{\mu_0}{4\pi}\int \frac{J}{r} dt'$, so $\nabla . A = 0$, $\nabla \times A = B$, and $\nabla . W = 0$ (We'll choose it so), $\nabla . W = A \Rightarrow W = \frac{1}{4\pi}\int \frac{B}{r} dt'$.

(b) W will be proportional to B and to two factors of r (since differentiating twice must recover B), so I'll try something of the form $w = \alpha r(r.B) + \beta r^2 B$, and see if I can pick the constants α and β in such a way that $\nabla . W = 0$ and $\nabla \times W = A$.

$$\nabla . W = \alpha\left[(r.B)(\nabla . r) + r.\nabla(r.B)\right] + \beta\left[r^2(\nabla . B) + B.\nabla(r^2)\right]$$

$$\cdot \nabla r = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3; \nabla(r.B) = r \times (\nabla \times B) + B \times (\nabla \times r) + (r.\nabla)B + (B.\nabla)r;$$

but B is constant, so all derivatives of B vanish, and $\nabla \times r = 0$ (Prob.1.62), so

$$\nabla(r.B) = (B.\nabla)r = \left(B_x \frac{\partial}{\partial x} + B_i \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z}) = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} = B;$$

$$\nabla(r^2) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{x} + 2y\hat{y} + 2z\hat{z} = 2r.$$

So

$$\nabla . W = \alpha [3(r.B) + (r.B)] + \beta [0 + 2(r.B)] = 2(r.B)(2\alpha + \beta)$$, which is zero if $2\alpha + \beta = 0$

$$\nabla \times W = \alpha [(r.B)(\nabla \times r) - r \times \nabla (r.B)] + \beta [r^2 (\nabla \times B) - B \times \nabla (r^2)] = \alpha [0 - (r \times B)] + \beta [0 - 2(B \times r)]$$

$$= -(r \times B)(\alpha - 2\beta) = -\frac{1}{2}(r \times B)$$

(Prob. 5.24). So we want $\alpha - 2\beta = 1/2$. Evidently a - 2(-2a) = 5a = 1/2, or a = 1/10; 0 - -2a = -1/5.

Conclusion: (But this is certainly not unique.)

(c) V x W = A /(V x W) - da = *f* A • da. Or / W • dl = JT A • da. Integrate around the amperian loop shown, taking W to point parallel to the axis, and choosing W = 0 on the axis:

,

(using Eq. 5.70 for A).

I*(s < R)*

$$W = \quad [1 + 2\ln(5/il)]\ z$$

Problem 5.54

Apply the divergence theorem to the function [U x (V x V)], noting (from the product rule) that V • [U x (V x V)] = (V x V) • (V x U) - U • [V x (V x V)]:

/v.[Ux(VxV)]dr = *J* {(V x V) • (V x U) - U • [V x (V x V)]} dr = flf [U x (V x V)] • da.

As always, suppose we have *two* solutions, B* (and Ai) and B_3 (and A2). Define $B_3 = B_2$ - Bj (and $A_3 = A_2$ - Ai), so that V x $A_3 = B_3$ and V x B_3 = V x Bj - V x B_2 = /ioJ - fM)J = 0. Set U = V = A_3 in the above identity:

J {(V x A3) • (V x As) - As • *[V* x (V x As))} *dr m j* {(B_3) | (B_3) - A_3 • (V x B_3]} dr = *j(B_3fdr*

= | [A_3**x**(Vx A_3)] • da = x B_3) • da. But either A is specified (in which case A_3 = 0), or else B is

specified (in which case B_3 = 0), at the surface. In either ease /(A_3 x Bg) • da = 0. So J(B_3)^{2}dr = 0, and hence Bi = B_a. qed

Problem 5.55

From Eq. 5.86, B_{tot} = ~(2cos0* + sm00). There-

47TT

fore B ■ r = B„(Z . r) - ^2cos* = *(B0* - cos*.

4 nr^8

This is zero, for all 0, when r = R, given by £$_0$

Evidently no field lines cross this sphere.

Problem 5.56

(a) J = -9 = *9*^.- *a* = 7rH^2; m = *^-tcR^2 z = %u>R^2* z. *L = RMv - MuR^2;* L = *Mu>R^2* z. (27r/w) 2?r 2ir 2

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MOTttQ

2?xR?

, or

R= fMomo\$^{1/s}$

n _ Q wR Q L* 2 Mwi?^{2} 2M"

— I&H.

and the gyromagnetic ratio is

9 =

2 M

(b) Because *g* is *independent of R,* the same ratio applies to all "donuts", and hence to the entire sphere (or any other figure

of revolution):

9

y v e n _ *eh* _ (1.60 x 1Q-^{19})(1.Q5 x IQ-^{34}) $^{L m}$ 2m 2 4m 4(9.11 x 10~^{31})

Q

2 M'

4.61 x 10~^{24} A m^2.

J(V x A)

/{/H

4»r/*a 4?r

(4HS J ^ X {j? * £*r" Note t*)at J depends on the source point r', not on the field point r. To do the surface integral, choose the (x, y, z) coordinates so that r' lies on the z axis (see diagram). Then * = y/R? + (*')² - 2Rz'cos0, while da = R² sin 0d$d<t>r. By symmetry, the x and y components must Integrate to zero; since the z component of t is cos0, we have

Problem 5.57 (a) Bavc =

1 1 ~~tw+im'U~~«f S-jsl ____ _

-Mr)* -7Rx'u*6 J0 y/Jp + (z'y -2Rz'co*0 161

Ustu «odii» -gin0d0.

tat fl , » f, v//P + (*')* - 2/k'u

2[2(rt² + (z')²) + 2i2z'd —— ---------------------- 1

3~~(2hVP~~ ----------V/P + W-W- J

= -~~g^p~~ {[iZ² + + Hz*] v/H² + (z')² -2tfz' - f/2² + -H^] ft] {[H² + (*>)² + Rz"\ \R - - [H² + - fbt\ (R + z*)}

'Air , „ 4tt . r—z z = —r, 3 3

4tril³ R², . . m

n

dm

d0.

For now we want r7 < fl,soBftve = J{3xt')dr' m J(Jxr')^. Now m . JJ(rx J)<fr

(Eq. 5.91), so Bave = qed

(b) This time r» > R, so Bw, = "M^f^ / ('1 * p?) ** = S / ^ ^ where *nOWgDe8

(a) Problem 5.51 gives the dipole moment of a shell: m = — owtf z. Let R-* r,a pdr, and integrate: 4tt a fR a . 4jt H S 0

4 . 4tt i*6 „ q

/k\ r» 2m

Po QvR² sin 0 ? 4tt 5 r²

(d) Use Eq. 5.07, with R —t f,cr pdf, and integrate: A paupsinO j fR _4 uquj 3Q sind R6 ?

A = 1" — U = __________________

This is *identical* to (c); evidently the field is pure dipole, for points outside the sphere.

(e) According to Prob. 5.29, the field is B = fl - cos0? - (l - s i n j . The average

(4*)²fl³ 3

from the source point to the *center* (-» = ~r'). Thus Bave — Bcen • qed Problem 5.58

m = -Qu>R² i. 5

fip 2Qu 4*r 5R '

4ir r

/xq Qu;i?² sin 0 ? 4tt 5 r²

Bdr

dr = x da —

(3/4)7T/?3 A x da =
4ttH3
3 n0
4ttB3 3/xo
obviously points in the * direction, so take the z component of * (cos0) and § (-sin0):

4/3)irR? $\int$

r2 sin 9drdBd<f>
Bm

5 R?)
4ttR (
_ lesS-a. f" \(i (4*&)* U IV 3 iSR'J cos'O + (4 - HI sin2 fij sinM i: 3 5 6HV

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Problem 5.59

The issue (and the integral) is identical to the one in Prob. 3.42. The resolution (as before) is to regard Eq. 5.87 as correct outside an infinitesimal sphere centered at the dipole. *Inside* this sphere the field is a delta-function, A5³(r), with A selected so as to make the average field consistent with Prob. 5.57:
3 . _ {Mj 2m _ 2^pm 4TT«3 4?r#3=* 3
m<5³(r).
. The added term is
Bnve —
Problem 5.60

(a) *IdX* -¥ Jdr, so
n=0

3 dr ■= *4irr dt* (Prob. 5.7), where p is the total electric dipole moment. In magne-
to *statics,* p is constant, so *dp/dt* — 0, and hence A_{mon} = 0. qed (c) m = /a = \ *If{r* x dl) -f m = | /(r x J) *dr*. qed
Problem 5.61

For a dipole at the origin and a field point in the *xz* plane *(<p* = 0), we have

^—*(2cos9t + sin9$)* = cos 0(sin *9* x + cos *9 z)* + sin 0(cos *9* x - sin *9 z))*

^[3sin0 cos0x + (2 cos² $ - sin² 0)«].
B ==
Here we have a *stack* of such dipoles, running from *z » -L/2 to z - +L/2.* Put the field point at *s* on the *x* axis. The x components cancel (because of symmetrically placed dipoles above and below *z* = 0), leaving B =
Po
dzy where *M* is the dipole mo-
4* 7o *rM* ment per unit length: *m m IirR² - (<rvh)%R² = auRnR²h =>*
sin
M m ~ as irawR. h*
-s cot 9 dz —
sin 0
1/2 (3 cos² 0 — 1)
i.
2Mz
1

Now sin0 = - r
dO. Therefore

obviously points in the * direction, so take the z component of * (cos0) and § (-sin0):

4/3)irR? J

r² sin 9drdBd<f>

Bm

5 R?)

4ttR (

_ lesS-a. f" \(i_ (4*&)* U IV 3 iSR'J cos'O + (4 - HI sin² fij sinM i: 3 5 6HV J

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Po

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4* 7o *rM* ment per unit length: *m m IirR² - (<rvh)%R² = auRnR²h =>*

sin

M m ~ as irawR. h*

-s cot 9 dz —

sin 0

1/2 (3 cos² 0 — 1)

i.

2Mz

1

Now sin0 = - r

dO. Therefore

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sin³ 0 5

HoauB? A /**

Mo

2 7T

/ (3 cos² 0 - 1) *Jwn*

B

= 2 I (3 cos² 0 — 1) sin 0 d0

s⁴ sin 0

uo*CTUJE? „ ,* *■ ʰ₋₂ z (- cos³ 0 + cos 0)

mO I If

, and cos 0,

+ PPr

cos 0m (l — COS 0m) z as

2*²

flQ<7L>R³L

B

4[s² + (L/2)²]³/²

COS0m sin 0m z.

2s² (1/2)

/2

But sin

z.

=, so

yV 1 (I/2)²

| Chapter 6

Magnetostatic Fields in Matter

Problem 6.1

N = m j X Bj; Bi = [3(mi • r)f - mi]; r = y;mi = m^; m₂ = m₂y. Bi =

= -^^x. Here m, = Tra²/, m₂ = *91*. So

Final orientation

downward] (—8).

Problem 6.2

rfF = /dl x B; dN = r X d F = / r x (d i X B). Now (Prob. 1.6): r x ((fl X B) + dlx (B X r) + B X (r x t f l) = 0. But d[r X (r X B)] = dr X (r x B) 4- r x (dr X B) (since B is constant), and dr = dl, so d l x (B x r) = r X (d X B) - d[r X (r X B)J. Hence 2r X (<fl X B) = d[r X (r x B) j - B x (r X dl). *dN* = *\I{d[r* x (r x B)] - B x (r x dl)}. N = fl {/d[r X (r X B)] - B X x dl)}. But the first term is zero (*f* d(- • •) = 0), and the second integral is 2a (Eq. 1.107). So N - -/(B x a) = m x B. qed

M

According to Eq. 6.2, *F = 2-kIRBcosO*. But B = and Bcosd _ B y, so *Bcos$* = p-ps [3(111! - r)(r • y) - (mi • $)]. But m₃ • y = 0 and f • y = sin *<f>;* while mi ♦ f = *m\ cos 6. BcosO* — ££ i3mi sin *<f>* cos *d>*

Problem 6.4

rfF - / {(dy 9) X **B**(0,y,0) + *(dxt)* **x** B(0,e,*) - *(dy9)* **x** B(0,y,c) - *(dzZ)* X B(0,0,*z*))

I\ ~(dy$) X [B(0, y,c)] - [B(0,y,0)]+(d*&) x [B(0,c,s)] - [B(0,0,] } TOx g - *f* X g} . [Note that /dy * e ^ and *Jdz* jf | * e *f*

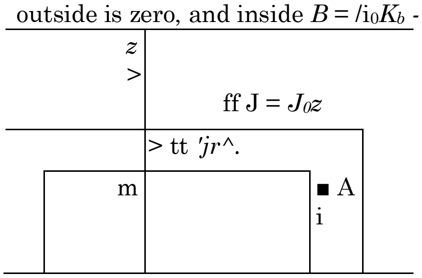
0,0,0
9 **1**
y o
x **0**
x 0
. » *dBx dBy BBt „dB,*
F = m
dBrn dBrn
dy dy ay
9~t
using V.B = 0 to write ^ + ^ =
dy dz dx
' 6 B X AdBx adBx
= m
But m • B = mB_x (since m = mx, here), so V(m ♦ B) - $m\,V(B_x)$ = *m* -f + Therefore F = V(m • B). qed
Problem 6.5

(a) B = £toJ0xy (Prob. 5.14).
m • B = 0, so Eq. 6.3 says F = 0.

(b) m • B = mofxoJoX, so F = mofioJoH.

(c) Use product rule #4: V(p • E) = p x (V x E) + E x (V x p) + (p • V)E + (E • V)p. But p does not depend on *(x,y,z),* so the second and fourth terms vanish, and V x E = 0, so the first term is zero. Hence V(p • E) = (p • V)E. qed
This argument does *not* apply to the magnetic analog, since VxB^O. In fact, V(m • B) = (m • V)B + no(m x J). (m • V)B0 = m0^(B) = mo/xoJby, (m • V)B6 = mo^fioJox?) = 0.
Problem 6.6
Aluminum, copper, copper chloride, and sodium all have an *odd* number of electrons, so we expect them to be paramagnetic. The rest (having an even number) should be diamagnetic. I
Problem 6.7

J6 - VxM = 0; Kfc = M x ft = M0.
The field is that of a surface current Kb = M0, but that's just a solenoid, so the field outside is zero, and inside *B* = /i0*Kb* - M. Moreover, it points upward (in the drawing), so **B = /<pM.**



Problem 6.8

$\nabla \times M$ s $J_{fc} = ks^2)i$ = i(3fca^2)S = pil Kj = M x n = $ks^2\{j>$ x s) = $-kR^2Z$.

So the bound current flows up the cylinder, and returns down the surface. [Incidentally, the *total* current should be zero ... *is* it? *Yes,* for $JJ_b da = f_0^R(3ks)(2irsds)$ = ggKg while $fK_b dl$ §J $(\sim kR^2)(2nR)$ 1 Since these currents have cylindrical symmetry, we can get the field by Ampere's law:

$B \cdot 27rs = fxoI_{Knc}$ — $pa\ I\ Jbdo.$ — $2ivkpos^3$ => B = $poks^2 4>$
JO ----------------

Outside the cylinder J_{enc} = 0, so | B = 0.

Problem 6.9
= p.oM.

mf ¥

115

K_t = M X n = $M4>$.

(Essentially a long solenoid)

(Essentially a physical dipole)

(Intermediate case)

[The external fields are the same as in the electrical case; the *interned* fields (inside the bar) are completely different—in fact, opposite ih direction.]

Problem 6.10

I fa - M_t so the field inside a *complete* ring would be poM. The field of a square loop, at the center, is givenby **Prob. 5.8: B,$_q$ =** $y/2\ pqI/tcR$. Here I = Mw, and R = a/2, so

$y/2poMw\ 2y/2PQMW$ _ _ ., . = = — n e t f i e l d m g a p :

2y/2t

B = poM 1 -

Problem 6.11 ' . :J |

As in Sec. 4.2.3, we want the average of B = B_{out} + B_{ln}, where B_{out} is due to molecules *outside* a *sm$* sphere around point and $B_{,n}$ is due to molecules *inside* the sphere. The average of B_{out} is same as field at center (Prob. 5.57b), and for this it is OK use Eq. 6.10, since the center is "far" from all the molecules k question:

M x i
_ *Afo f* 47X J
outside

dr
*out

The average of Bin is g Eq. 5.89—where ra = ±irR³M. Thus the average B_{in} is 2^qM/3. But what is *left out* of the integral A_{out} is the contribution of a uniformly magnetized sphere, to wit: 2^M/3 (Eq. 6.16), and this is precisely what B_{jn} puts back in. So we'll get the correct macroscopic field using Eq. 6.10. qed .

Problem 6.12

(a) M = *ksz* \ J_b - V x M = K_b = M x n = *kR<j).*

B is in the *z* direction (this is essentially a superposition of solenoids). So B = 0 outside. Use the amperian loop shown (shaded)—inner side at radius *s:*

f B - dl = *Bl* = /i04nc = Po [M *da* + $K_b l$] = jz0 *[~kl(R -$) + kRl] - yokh.*

B = noksz. inside.

(b) By symmetry, H points in the *z* direction. That same amperian loop gives *fH* • *di = HI - VoIf*$_{ne}$ - 0, since there *is* no free current here. So | H = 0 , and hence B = /i0M. *Outside* M = 0, so B = 0; *inside* M = ksz, so B = {lohsz.

Problem 6.13

(a) The field of a magnetized sphere is |/i0M (Eq. 6.16), so with the sphere removed.

In the cavity, H = ^B, so H = ~ (B0 - f/i0M) = H0 + M - |M

The field inside a long solenoid is ^K. Here K = M, so the field of the bound current on the inside surface of the cavity is /ioM, pointing *down.* Therefore

B = Bo - MoM;

H = —(Bo - /x0M) = —Bo - M => H = Ho.

Kb This time the bound currents are small, and far away from the center, so B = Bp, |
while H = j~Bo = H0 + M | H = H0 + M.

[Comment: In the wafer, B is the field in the medium; in the needle, H is the H in the medium; in Dm sphere (intermediate case) both B and H are modified.]

B = Bo - ^MoM,

H = Hn + ^M.

(b)

(c)

- M.

M

\ $\boxed{2 \quad \text{-/ioM.}}$

Problem 6.15

1Potentials

W$_{in}$(r,0) =]T *Air¹ P$_l$(cos6),* (r < *R); W$_{out}$(r,$) = (r>R).*

Boundary Conditions:

J (i) *W$_{ia}${R, 9) = W$_{ollt}$(R,9),* 100 -^r \ $_{Jt}$ + ^ \ $_R$ = *M^* = *Mz.r=Mcose.* {The continuity of *W* follows from the gradient theorem: W(b) - PF(a) = *VW* - dl = - H • dl; if the two points are infinitesimally separated, this last integral 0.)

V *A$_t$R^l* — = *R^{2l+l}A$_h$*

V *Z{l + l)$_1$lfaP$_l$(casO) + ZlAtR?-*P$_i$(cose) = Mcos0.*

^Combining these:

1 + l)R^l~^{l}AtPtica&O) - M cos 0, so *At* = 0 *(I ?* I), and 3A$_X$ = *M* =» Ai =

Thus WjnM) = yr cos fl' = —and hence H_{ln} = -VW$_{in}$ = -y 2 = —M, so

B = /io(H + M) = /io f- ^M + M^ =

Problem 6.16

/H-dl = //.,„, =/, so H = B = Al0(l + Xm)H =

Jfc = VxL. — _ , « K, 5 *ds \ 2l(8 J* 168

Total enclosed current, for an amperian loop between the cylinders:

1 + 2^27r° + Xm)I> 80 f8 " S = | Ml + pi f B = /

Problem 6.17

Prom Eq. 6.20: $H • dl = *H(2ns) = I* = S*^1^^ ^ < a); *11* *(s > a).*

Mt' iiit itsi!

Jt - *XmJf* (Eq. 6.33), and *Jj* - so K_{fc} = M X n = XmH x n

Xml 2 ira

M = XmH = *Xml 2 2m?'*

= M x n = f^fz, ats = a; [- ^fz, at r = 6.

1 ani I 2m'

(s>o).

J = Xm/ 6 ?ra²

(same direction as I),

Kfc =

(opposite direction to I).

Ib = Jb(na?) + Kb(2na) — *Xml* ~ ym/ = |o| (as it *should* be, of course).

Problem 6.18

By the method. of Prob. 6.15:

For large r, we want B(r,0) Bo = B_0z, so H = j^Boi, and hence *W -> -j^Boz - -jfcBorcosO.*

Potentials

■W_n(r,0) « £v4,r'P,(cos0), (r < *R);*

BWout(r,^) = — */fc-Bor cos 0 + J2* pqfr **P**j (cos **0),** (*r > R* **).** *Boundary Conditions:*

JO) Wi,,(R,#) = Wmit(fl,»), i(«) -^Mk + ^L-o. (The latter follows from Eq. 6.26.)

^$H_o$cos0 + |gg + 1) Pi (cos # j^ M/^^cosfi) = 0.

For i ^ 1, (i) *Bi* = so + 1) + /iiJA/i?""1 = 0, and hence *At* = 0.

For / = 1, (i) => *AiR = —fcBoR+BJR², * and (ii) =» *B₀ + 2miBi/R* + nAₓ* = 0, so *Aₓ*** -3B_0/(2w»t^

(ii) =S> Ho

WW*'* = "(2^)

B = /iH =

3B_0

3Br

ZBqz **(2po + p)'**
3/iBo

Hln = -Wm =
■ z =
rcosf = -
(2po + M) (2/IQ + #}'
i+Xm \ R
ITwaJ®0-
(2/UO + M)

By the method of Prob. 4.23:
■ *Step 1*: B_0 magnetizes the sphere- M f sphere given by Eq. 6.16: ~~ ~ ~~Mo(f+xm)~~B°- This magnetization sets up a field 169 within

Bl & 3<*M° **- flf^Bo = §«B_0** (where « | | J the sphere- Bl magnetizeS the sphere an additional amount M_x = £B_a. This sets up an

additional field in

B_2 = l^oMj * ?«Bi « ^yl B_0, etc.

The *total* field is:

B = B_0 4- Bi + B_2 + • • • • = B_0 + (2K/3)B_0 + (2K/3)$^2B_0$ + •., = [!+■ (2*/3) + (2/c/3)^2 + • • •] B_0 = B°
(1 - 2K/3)

1 3 3 + 3xm 3(1+ Xm)
SO
1 — *2k/Z Z-2Xm /(l+Xm)* **3 +** *3Xm* **- 2Xm 3 + Xm**

Problem 6.19

eªr* r». A/r _ Am _
Am = -f^-B; M = ^t^2 = ~4^7vB' where v is the volume per electron. M = *$_m$H (Eq. 6.29) = ~~BMW~~b (Eq. 6.30). So Xm = "^fMo-
[Note: Xm 1, so I won't worry about the (1 + Xm) term; for the same reason we need not distinguish B from B_{else}> as we did in deriving the Clausius-Mossotti equation in Prob. 4.38.] Let's say *V* = §7rr^3. Then Xm = ru use 1 A= 10~^{10} m for r.
Then Xm = — (10~^{7}) (~~4(9^3^f0-9'i)(T(j-»)~~) = ~2 x 10"M which is not ^d~Table 6.1 says Xm = -1x 10
However, I used only *one electron* per atom (copper has 29) and a very crude value for r. Since the orbital radius is smaller for the inner electrons, they count for less (Am ~ r^2). I have also neglected competing paramagnetic effects. But never mind ... this is in the right ball park.

Problem 6.20

> Place the object in a region of zero magnetic field, and heat it above the Curie point—or simply drop it on a hard surface. If it's delicate (a watch, say), place it between the poles of an electromagnet, and magnetize it back and forth many times; each time you reverse the direction, reduce the field slightly Problem 6.21

V Identical to Prob. 4.7, only starting with Eqs. 6.1 and 6.3 instead of Eqs. 4.4 and 4.5.

V Identical to Prob. 4.8, but starting with Eq. 5.87 instead of 3.104.

V *U = -g^r[3cos0i cos#2 - cos(0$_2$ - 0i)]mim$_3$. Or, using cos(0$_2$ - 0i) = cos0i cos0$_2$ - sin0i sin0$_2$,*
U = — ~~ff$^{1*^}$~~ (sin sin 0_2 — 2cos0i cos0_2).

------- ------------------- --------- -------- lyj. d&i – **g02 — w**

w _ ~~My~~ (cosi9l sin0_2 + 2sin^i cos0_2) =0=> 2sin0i cos0_a = - cos0i sin0_2; fg = ~~^ff*~~ (sin 9\ cos + 2 cossin0_2) = 0 2 sin 0a cos0_2 = - 4cos0i sin h
the

------ - qovm* v»jr uij. u,xg;

Stable position occurs at minimum energy: = = 0

k®

Either sin0i = **sin0₂ = 0 : —>~—► or or cos#i** p cos 02 — 0 **: ft or t**

Which of these is the *stable* **minimum? Certainly not© or®—for these m₂ is no parallel to Bj,** whereas *m* know m₂ will line **up along** Bj. **It remains to compare© (with Q\ = 02 = 0)** and® **(with 6\ — tt/2,** 0₂ = -jr/2): U_x - **M^jJr(~2);** *Ui* - **1)**
Ui **is the lower energy, hence the more stable configuration.**

Conclusion: They line **up parallel, along the line joining them:**

(d) They'd line up the same way:

Problem 6.22

F = /^ca x B = * *Ba+I £ dLx[(v* V0)B0] ~ I (J> dI) x Kr° * v0)B0] = *Ijd* x **[(r • V0)B0J**

(because fdl - 0). Now

(di x Bo)* = *€ijkdlj(B0)k,* and (r • V0) = so

j> ti dlj [(V0)i(Bb)jfe] ^ Lemma 1: *j> n dlj* = **X)** (proof below), **j jj |**

= *I £ijJ#iS(Vo)/(Bo)k* | Lemma 2:]T *eijkttjm - Suhm - hm&ki* (proof below). 1 **= / J2 - MVo)i(£o)fc = /** *Y,*

*MV0)i(Boh - *i{VoMBo)k}*

k,l,m *k*

= J[(Vo)i(a ■ Bo) — a,(Vo • Bo)]. But V0 • B0 - 0 (Eq. 5.48), and m = /a (Eq. 5.84), so **F =** V0(m • B0) (the subscript just reminds us to take the derivatives at the point where m is located), qed *Proof of Lemma 1:*

Eq. 1.108 says /(c • r) *dl* - a x c =-*c* x a. The jth component is *f* W *dh* = ~ EP,m cjpmCpOm- P»ck *Cp — 5pi* (i.e. 1 for the Zth component, zero for the others). Then / « *dlj* - ~ €jim«m = Em £'im<lm' ^ *Froo/ o/ Lemma 2:*

must be the other, so

^tijktijm = + *BS*im*Skt-*

To determine the constant pick t = i = 1, fc = m = **3;** the only contribution comes from j » 2:

€123^123 = 1 = -4^11^33 + *£613631* = j4 => ^ =

To determine B, pick **t** = m = **1,& = / = 3:**

€1236321 = "I = ^13*31 + Htl^ = ^ =» B » -1.

J^ cyfccym - - *6im6u-* ^

Ft = I

»/ o/ Lemma 2:

= 0 unless *ijk* **and /jm are both permutations of 123. In particular, « must either be** *I or* **m, and 9**

So

Problem 6.23

in8ideaUnif°rmiy *P°larized* sP^re, E = **-g—P** (Eq. 4.14) **translates toH =** *-fcipoM)* **= _J—• 'u - Mo(H-fM). So the** *magnetic* **field inside a uniformly** *magnetized* **sphere is** B = po(- |M+M) =
| gftoM j (same as Eq. 6.16).

(b) The *electric* field inside a sphere of linear *dielectric* in an otherwise uniform *electric* field is E = ~~v+x./s~~^*0 (Eq. 4.49). Now translates to Xm, for then Eq. 4.30 **(P** = e0xeE) goes to poM = ^^mH, or M = *mH (Eq. 6.29). So Eq. 4.49 H == ~~1+ym/~~3HQ. But B = /xo(l + *ro)H, and B0 = ^0H0 (Eqs. 6.31 and 6.32), so the *magnetic* field inside a sphere of linear *magnetic* material in an otherwise uniform *magnetic* **field** is B 1 171
Bo
(as in Prob. 6.18).
/*o(l + Xm) (l+Xm/3)/i0'

(c) The average *electric* field over a sphere, due to charges within, is Eave == -jsfcy Let's pretend the charges are all due to the frozen-in polarization of some medium (whatever *p* might be, we can solve V-P — —*p* to find the appropriate P). In this case there are *no* free charges, and p = *JP dr*, so Eave = -4™ 753- *fPdr,* which _^anslatesto ; ^
'poM dr =—
4irR3
1 + Xm 1 + Xm/3
Bo
B =
Have —
But B = *po{H* 4- M), so Bave = + /JoMave, and Mave = so

with Eq. 5.89. (We must assume for this argument that all the currents are *bound*1 but again it doesn't really matter, since we can model any current configuration by an appropriate frozen-in magnetization. See G. H. Goedecke, *Am. J. Phys.* 66,1010 (1998).)
Problem 6.24
Eq. 2.15 : E = *p fv * dr'}*
Eq. 4,9: V «
Eq. 6.11: A a *p0e0U* x { ^ *fv dr>}*
_ **f**
. For a uniformly charged sphere (radius R): <
I Eout
in agreement

So the scalar potential of a uniformly polarized sphere is:

= ^(Mxr), Aout « ^£?(Mxf),
and the vector potential of a uniformly magnetized sphere is:

(confirming the results of Ex. 4.2 and of Exs. 6.1 and 5.11). Problem 6.25
(a) Bi « (Eq. 5.86, with *0* = 0). So m^Bj = -gj. F = V(m-B) (Eq. 6.3) =» F = £ [-

This is the magnetic force *upward* (on the upper magnet); it balances the gravitational force downward *[-mdgi):*
- *m<t9* 55 0

	Zpom²	
X	2it	
38	m<t9.	

po **2m**

B»ve —

4?r R³ '

(for uniform charge density); (for uniform polarization); (for uniform magnetization). 53 (Prob. 2.12),

= KsMf*) (Ex. 2.2).

Vout «▶

A

Mo m* 1 a

3/ip mª 2πz*

(b) The middle magnet Is repelled *upward* **by lower magnet and** *downward* **by upper magnet:**

Zp0m² ZpQm²

The top magnet is repelled *upward* **by middle magnet, and attracted** *downward* **by lower magnet:**

3 Atom² 3/xom² .

Subtracting: [J, - J, - J, + -$m_d g$+$m_d g$ *m* **0, or** **« 0, so: 2 = ^+^g**

Let q = *x/y,* **then 2 = ^**r **+ ~~i_tt+|)4~~ - Mathematica gives the numerical solution** *a* = *x/y* **= 0.850115...**

Problem 6.26

At the interface, the perpendicular component of B is continuous (Eq. 6.26), and the parallel component of **H** is **continuous (Eq. 6.25 with K/ = 0). So** *B{-* **=** *Bfr,* **H$ = H|. But B =** *pH* **(Eq. 6.31),** so = **^Bf. Now tan^i =** *B\jB^* **and tan0₂ =** *B\lB£,* so

h **I***S?£* **IH «***&*

tan 0i *BI* **b**JI *PI*

(the same form, though for different reasons, as Eq. **4.68).**

Problem 6.27

In view of Eq. 6.33, there is a *bound* dipole at the center: m_6 = So the *net* dipole moment at the | center is m^nter = m + m_b = (1 + Xm)m = j^m. This produces a field given by Eq. 5.87:

Bcenter T~""T [3(m • f)f - m l. dipole 4?r r³ ' 1

This accounts for the *first* term in the field. The remainder must be due to the bound surface current (K*) at I *r* ~ *R* (since there can be no volume bound current, according to Eq. 6.33). Let us make an educated guesi 3 (based either on the answer provided or on the analogous electrical Prob. 4.34) that the field due to the surface I bound current is (for interior points) of the form $B_{gurface}$ — -4m (i.e. a constant, proportional torn). In that |

« . . .„ , current

case the magnetization will be:

M = XmH = —B = [3(m • f)r - m] + ^Am *p 4 n r ** *p*

This will produce bound currents J_b = VxM = 0, as it should, for 0 < r < *R* (no need to *calcuUU* thiti curl—the second term is *constant,* and the first is essentially the field of a dipole, which we know it curl-Ism* i except at r = 0), and

K_6 = M*(R)* X *t m* ^(-m xf)+ xf)«x-m (-4^5 + ^)

But this is exactly the surface current produced by a spinning sphere: K - a v = awR sin 9+t with (owA) flS Xn>m - j^RtV So the field it produces (for points inside) it (Eq. 5.68):

Bturfoce « *Zpo(ou>R)* ® *ZpQXm*** (~ - r-sr 1 current 3 3

122
tan

Everything is consistent, therefore, provided *A* = §MoXm - *or A* (l - ^g**Xm*J - *But*
x„ = -1, so A (l - I + |f) = or *A* (l + = A = and hence 173

I + **I***

The *exterior* field is that of the central dipole plus that of the surface current, which, according to Prob. **5.36,** is *also* a perfect dipole field, of dipole moment

w _ msurface ~ current **3** _ **4** *3* / **3** *\auJK)* — *-TTtt* I —^surface J ~ **3 \2/Zq** current/ *fiQ 4lT* «J(2/io + /*) A»o(2/4) + *fl)* ^ _ *2irR^z p.* **2(/ip -** *fi)m* _ **/ifcp -** *fi)m* *"T~* r>a/n — i—\ —To

So the *total* dipole moment is:

™ - ** ~ . M _ (Po - M) 3/um mtot — —m H m-
--- ui T ------ 111 . v —' /-A • -■ • •■■ \ ,
Pq /Xo (2/io + W + At)

and hence the field (for r > *R)* is

^ 4tr

Problem 6.28

The problem is that the field inside a *cavity* is not the same as the field in the material itself.

V *Ampire type. The field deep inside the magnet is that of a long solenoid, Bo « /ioM. From Prob. 6.13: (Sphere: B = B0 - f^oM s | /*oM;*

< Needle : B = B0 - fi0M = 0; [Wafer : B = /10M.

(b) *Gilbert type. This is analogous to the electric case. The field at the center is approximately that midway between two distant point charges, B0 « 0. From Prob. 4.16 (with E B, 1/eo Ho, P M):*

(Sphere : B = B0 + *fM m*

< Needle: B = B0 = 0;

(Wafer : B = B0 + MoM = *FIOM.*

h»the cavities, then, the fields are the *same* for the two models, and this will be no test at all. Yes. Fund it ***with*** **tl M** from the Office of Alternative Medicine.