Proposition 9. Let A be a left G-module and M a Z-module; further, let A ®z M be given the structure of a left G-module for which cr(a (g) x) = <ra ® x (aeA, aeG, xeM). Checked 25/2 9:08	Định đề 9. Giả sử A là một G- mô-đun trái và M là một Z-mô-đun ; thêm vào đó, giả sử A $\otimes_Z$ M là một cấu trúc nào đó của G- mô-đun trái có tính chất $\sigma(a\otimes x)=\sigma a\otimes x \quad (a\in A, \sigma\in G, x\in M).$ Proposition: cũng có nghĩa là "mệnh đề"
If now A is G-special then so is A ®ZM. In particular, Z(G) is G-special.	Bây giờ, nếu A là G-đặc biệt thì A $\otimes_Z$ M cũng vậy. Đặc biệt, Z (G) là G-đặc biệt.
Proof. Let u : A -+A be a Z-homomorphism such that a = E <r{w(<r_1a)} <T for all aeA and put v = u ® iM, where iM is the identity map of M. Then, ifae.4, xeM and y = a®x, 2 (T\V{O 1y)} = 2 o-{u(cr-1a) (g) a} = £ {^(a—xa) ® x} = a <g) a; = y, <r a <r and the proposition follows immediately.	Chứng minh. Giả sử u : A →A là một Z- đồng cấu sao cho $a=\sum_\sigma \sigma\{u(\sigma^{-1}a)\}$ đối với mọi $a\in A$ và đặt v = u⊗i_M , trong đó i_M là ánh xạ đồng nhất của M. Thế thì, nếu $a\in A$, $x\in M$ và y = a ⊗ x , $\sum_\sigma \sigma\{v(\sigma^{-1}y)\}=\sum_\sigma\sigma\{u(\sigma^{-1}a)\otimes x\}=\sum_\sigma\{\sigma u(\sigma^{-1}a)\otimes x\}=a\otimes x=y,$ và chúng ta sẽ suy ra được định đề ngay lập tức.
There is another result, complementary to the one just proved, which can be stated thus:	Ngoài ra còn có một kết quả khác, bổ sung cho kết quả vừa được chứng minh ở trên, có thể được phát biểu dưới dạng như sau:
Proposition 10. Let Bbea right G-module and M a Z-module; further, let Homz (B, M) be given the structure of a left G-module in which, for /e Homz(B, M), (^)6 =y(6(r) (6€5> (T€G)	Định đề 10 . Giả sử B là một G- mô-đun phải và M là một Z-mô-đun ; Thêm vào đó, cho Homz (B , M) là một cấu trúc nào đó của G- mô-đun trái trong đó , đối với f ∈Hom_z(B , M), $(\sigma f)b=f(b\sigma)\quad(b\in B,\sigma\in G).$
If now B is G-special, then so is Homz (B, M). In particular, Homz (Z(G), M) is G-special.	Nếu bây giờ B là G- đặc biệt , thì Hom_z(B , M) cũng vậy. Đặc biệt, Hom_z (Z (G) , M) là G- đặc biệt.
Proof. Let u : B->B be such that b = 2 {u(ba~1)}(T <r for all beB, and put v = Horn (u,iM). If now/e HornZ(B, M), then U <T~lf v{(T-1f) is the combined mapping B->B-> M, consequently, if we write <r{«(<r-1/)} = (j)^, we shall have	Chứng minh . Giả sử u : B→ B sao cho $b=\sum_\sigma\{u(b\sigma^{-1})\}\sigma$ Đối với mọi b∈B , và đặt v = Horn (u, i_M) . Nếu bây giờ f ∈ $Horn_Z$ (B , M), thì $v(\sigma^{-1}f)$ là một ánh xạ kết hợp $B\xrightarrow{u}B\xrightarrow{\sigma^{-1}f}M$, do đó , nếu chúng ta viết $\sigma\{v(\sigma^{-1}f)\}$, chúng ta sẽ có

fa(b) = {«(cr_1/)} (bcr) = {tr-1/} «(6<r) = f{u(b<r) o-1}.
Accordingly (£ fa) b = f(b),
<r

and therefore 2 a’{v{(T~1f)} = Hfa=f'
O' (T

This completes the proof.
It is now possible to give two very useful criteria for a (?-module to be special. Let A be a left G-module, then Z(G) ®ZA has a structure as a left (?-module in which
tr(A ® a) = trA 0 a (AeZ(G),aeA, creCr). (10.13.5)
Further, there is a homomorphism
Z(G) ®ZA ->A

in which A®a->Aa, (10.13.6)

and this is clearly a Cr-epimorphism.
Proposition 11. Let A be a left G-modvle. Then A is G-special if and only if the mapping Z(G) ®z A ->A, defined in (10.13.6), is directf when regarded as an epimorphism of G-modules.
Proof. If Z(G) ®ZA -+A is direct then A is isomorphic to a direct summand of the Cr-module Z(G)®zA. But Z(G)®zA is (?-special (Proposition 9) hence, by Proposition 8, A is G-special.
To prove the converse, let A be G-special then the identity map iA of A is the norm of a 2-homomorphism u : A-+A. Let u* be the Z-homomorphism A->Z(G)®zA defined by a-> 1 ®u(a), then u : A ->A is the combined map
AXA\Z(G)®ZA^A.
Takingnorms and applying (10.13.4),

$$\phi_\sigma(b) = \{v(\sigma^{-1}f)\}(b\sigma) = \{\sigma^{-1}f\}\,u(b\sigma) = f\{u(b\sigma)\,\sigma^{-1}\}.$$

Vì vậy

$$\left(\sum_\sigma \phi_\sigma\right) b = f(b),$$

và do đó

$$\sum_\sigma \sigma\{v(\sigma^{-1}f)\} = \sum_\sigma \phi_\sigma = f.$$

[redacted]

$$\sigma(\lambda\otimes a) = \sigma\lambda\otimes a \quad (\lambda\in Z(G),\, a\in A,\, \sigma\in G). \qquad (10.13.5)$$

[redacted]

$$Z(G)\otimes_Z A\to A$$

[redacted]

$$\lambda\otimes a\to\lambda a, \qquad (10.13.6)$$

[redacted]

[redacted]

[redacted]

$$A \xrightarrow{i_A} A \xrightarrow{u^*} Z(G)\otimes_Z A\to A.$$

[redacted]

we find that iA can be represented as N(u*) A >Z{G) ®ZA>A,	[redacted] $A \xrightarrow{N(u^*)} Z(G) \otimes_Z A \longrightarrow A,$
and, since Nu* is a Cr-homomorphism, this completes the proof.	[redacted]
For the second criterion we use Hornz(Z(G),A) and endow it with the structure of a left Cr-module in which of, where /e Homz (Z(G),A),	[redacted] $f \in \mathrm{Hom}_Z(Z(G), A),$
is given by (<r/)A =/(A<r) (AeZ(G), o'e.G) (10.13.7)	[redacted] $(\sigma f)\lambda = f(\lambda\sigma) \quad (\lambda \in Z(G), \sigma \in G).$ (10.13.7)
Now, if a e A, the mapping A->Aa is a 2-homomorphism of Z(G) into A. Denoting this homomorphism by/, we have	[redacted]
/(A) = Ao, (10.13.8)	$f(\lambda) = \lambda a,$ (10.13.8)
and then a ->/ is a homomorphism A->'R.omz{Z{G),A), (10.13.9)	[redacted] $A \to \mathrm{Hom}_Z(Z(G), A),$ (10.13.9)
which one easily verifies is a <?-monomorphism.	[redacted]
f See section (1.9).	
Proposition 12. Let A be a left O-module. Then A is G-special if and only if the mapping A^-Homz (Z(G), A) of (10.13.9) is direct, when regarded as a monomorphism of G-modules.	[redacted]
Proof. If the monomorphism is direct, then A is isomorphic to a direct summand of the (5-module Homz (Z(G), A). But this is G-special (Proposition 10) consequently A is G-special.	[redacted]
Assume next that A is Cr-special and let the identity map iA of A be the norm of u : A ->A. Now the Z-homomorphism	[redacted]
u* : 'Komz(Z{G),A)-^A	$u^* : \mathrm{Hom}_Z(Z(G), A) \to A$
defined by u*{f) = /(1), is such that u	[redacted]

is the combined mapping
A -> Homz (Z(G), A) ”+ A -t A;

consequently, taking norms, iA can be represented as
Nu*
A >- Homz (Z(G), A) A
and, since Nu* is a 6r-homomorphism, this shows that
A^Tlomz(Z(G),A)
is direct.
As an application of the last two results we shall prove
Theorem 17. If A is either G-projective or G-injective, then A is G-special.
Proof. If A is Cr-projective then the epimorphism Z(G) ®ZA -+A of Proposition 11 is direct by Theorem 1 of section (5.1). On the other hand, if A is 6?-injective then the (?-monomorphism
A -> Homz (Z(G), A),
which occurs in Proposition 12, is direct by virtue of Theorem 6 of section (5.2). The theorem now follows.
10.14 Properties of the complete derived sequence
We are now in a position to establish some further facts about the complete derived sequence
J~*(G,A), J~1(G,A), J°(G,A), J*(G,A), ...
of an arbitrary finite group G. These additional facts stem from
Theorem 18. If A is G-special, then Jn{G,A) = 0 for all values ofn.
Proof. Let u : A -+A be a Z-homomorphism such that
« = 2 {<TM((T_1a)} tr
for all a e.4. If now aeA° then M((r-1a) = w(a) and therefore a is the norm of u{a). Thus Aa c N(A)

$$A \to \mathrm{Hom}_Z(Z(G),A) \xrightarrow{u^*} A \xrightarrow{i_A} A;$$

[redacted]

$$A \longrightarrow \mathrm{Hom}_Z(Z(G),A) \xrightarrow{Nu^*} A$$

[redacted]

$$A \to \mathrm{Hom}_Z(Z(G),A)$$

[redacted]

[redacted]

$$A \to \mathrm{Hom}_Z(Z(G),A),$$

[redacted]

[redacted]

[redacted]

[redacted]

$$\ldots, J^{-2}(G,A), J^{-1}(G,A), J^{0}(G,A), J^{1}(G,A), \ldots$$

[redacted]

[redacted]

$$a = \sum_{\sigma} \{\sigma u(\sigma^{-1}a)\}$$

[redacted]

consequently, by (10.12.13), it follows that J°(G,A) = 0. Assume next that aeNA then Na = 0 and therefore 2 u(cr-la) = u{Na) = 0. <r Accordingly a = 2 (^ — 1) {«(cr-1a)} e I A, <T hence NA c IA and so J~1(G,A) = 0 by (10.12.14). Put A' = ~Komz{Z(G),A) and let A' have the structure of a left G-module as indicated in (10.13.7). By Proposition 12, A is isomorphic to a direct summand of the G-module A' and, by Proposition 3, Jn{G,A') = Hn(G,A') = 0 for all n > 1. It follows that Jn(G, A) = 0 for 1. Finally, let A* = Z(G) ®ZA, where A* has the structure of a left Cr-module described in (10.13.5). Then, by Proposition 11, A is iso-morphic to a direct summand of A * (as G-module); furthermore, when n ^ 2, Proposition 4 shows that J^(G,A*) = Hn_1{G,A*) = 0. Accordingly «7~n(G, A) = 0 for w ^ 2 and with this the proof is complete. Lemma 1. Let / : A'A be a homomorphism of left G-modules which is the norm of a Z-homomorphism u : A' -*A. Then there exists a G-homomorphism A' ->Z(G) ®ZA such that f is the combined mapping A' —>Z(G) ®z A —>A. In this lemma, Z(G) ®ZA is to have the same structure as G-module and Z(G) ®ZA is to be the same G-homomorphism as in Pro¬position 11. Proof. Let u* : A' Z(G) ® z A be the Z-homomorphism defined by u*(a') = 1 ® u(a'), then u is the combined	[redacted] $\sum_{\sigma} u(\sigma^{-1}a) = u(Na) = 0.$ [redacted] $a = \sum_{\sigma} (\sigma - 1)\{u(\sigma^{-1}a)\} \in IA,$ [redacted] [redacted] $J^n(G, A') = H^n(G, A') = 0$ [redacted] [redacted] $J^{-n}(G, A^*) = H_{n-1}(G, A^*) = 0.$ [redacted] [redacted] [redacted] [redacted]

mapping *4' w* A'->A'^Z(G) ®ZA^A.	$A' \xrightarrow{i_{A'}} A' \xrightarrow{u^*} Z(G) \otimes_Z A \to A.$
The required result now follows from Proposition 7 on taking norms. Proposition 13. Let f : A'->A be a homomorphism of left G-modules and suppose that f is the norm of some Z-homomorphism of A' into A. Then Jn(G,f) = 0 for aU values of n.	[redacted]
Proof. Jn(G,f) is the homomorphism Jn(G,A')—*Jn(G,A) induced by/consequently, by Lemma 1, this can be represented in the form	[redacted]
Jn(G,A') -+Jn(G, Z(G) ®z A) -*Jn(G,A).	$J^n(G, A') \to J^n(G, Z(G) \otimes_Z A) \to J^n(G, A).$
But, by Theorem 18, the second term is a null module because Z(G) ®ZA is G-special (Proposition 9). The result follows.	[redacted]
Now let A be any G-module and let iA be its identity map. Since iA is a G-homomorphism, it follows that N(iA) = qiA, where q is the order of G. However Jn is an additive functor and so	[redacted]
Jn(G,qiA) = qJn(G, iA).	$J^n(G, qi_A) = qJ^n(G, i_A).$
This establishes the next theorem if one takes account of Proposi-tion 13. Theorem 19. Let G be a finite group of order q and A any left G-module. Then qJn{G, A) = 0 for all values of n.	[redacted]
Taking account of Theorems 17 and 18 we may observe that, inter alia, the complete derived sequence of G has the following properties: (a) the Jn(G,A) form an exact, connected sequence of covariant functors;	[redacted]
(b) J°(G, A) is the functor	[redacted]

consisting of the fixed elements of A modulo the elements which are norms;

(c) Jn(G,A) = 0 for all n whenever A is either G-projective or G-injective.

These suffice to characterize the sequence to within an isomorphism of connected sequences. Indeed, one has the following more general uniqueness criterion which will be needed later.

Proposition 14. Let A be a variable left G-module and let

..., T~\A), T~\A), T»(A), T^A), ...

and ..., U~2(A), U-HA), U\A), IP(A), ...

be exact connected sequences of covariant functors of A whose values are Z-modules. Suppose further that whenever A is either G-projective or G-injective, then Tn(A) = 0 and Un(A) = 0 for all values of n. If now, for a particular integer r, there exists a functor equivalence Tr{A) Ur{A), then this equivalence has a unique, extension to an isomorphism of the connected sequences.

The proposition needs no proof since it follows at once from the corollaries to Theorems 10 and 12 of section (6.5).

10.15 Complete free resolutions of Z

The method of obtaining the complete derived sequence of 0, by combining together the homology and cohomology theories, has the advantage of showing how these all tie up with one another; but it is inconvenient in that the two halves of the sequence are then on different footings and therefore tend to require separate discussion. In a moment we

[redacted]

[redacted]

[redacted]

[redacted]

$$\ldots,\ T^{-2}(A),\ T^{-1}(A),\ T^{0}(A),\ T^{1}(A),\ \ldots$$

[redacted]

$$\ldots,\ U^{-2}(A),\ U^{-1}(A),\ U^{0}(A),\ U^{1}(A),\ \ldots$$

[redacted]

[redacted]

[redacted]

shall describe a method by which this can be overcome, but first, in order not to interrupt the main development at an awkward moment, we shall establish a property of exact complexes of Z-free modules.

Proposition 15. Let T(M) be an additive functor of Z-modules, whose values are also Z-modules, and let X be an exact complex

whose component modules are Z-free. Then T(X) is also exact.

Proof. We shall suppose, for definiteness, that T is a covariant functor. The contravariant case can be treated similarly. Put

Im (Xn —> = An

then, for each value of n,

0->J4n+1->Xm->^4TC->0 (10.15.1)

is an exact sequence. Now An is a submodule of the Z-free module Xre_1 and Z is a principal ideal domain, consequently, by Theorem 3 of section (9.1), An is also Z-free. It follows that the exact sequence (10.15.1) splits and therefore, since T is additive,

0 -»■ T (An+1) -+T(Xn)->T(An)->0

is exact for all n. But T(Xn+1) -> T(Xn) and T(Xn) -> T(Xn_^) can be represented by

T(Xn+1)^T(An+1)->T(Xn) and T(Xn)-^T(An) -> T(Xn_1) respectively and now it can be seen that

nx^j^nxj^nx^)

is exact as required.

We come now to a new concept. Regarding Z as a left G-module (on which 0 acts trivially), we define a complete G-free resolution of Z as a pair of exact sequences

[redacted]

[redacted]

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

[redacted]

$$\mathrm{Im}\,(X_n \to X_{n-1}) = A_n$$

[redacted]

$$0 \to A_{n+1} \to X_n \to A_n \to 0 \qquad (10.15.1)$$

[redacted]

$$0 \to T(A_{n+1}) \to T(X_n) \to T(A_n) \to 0$$

[redacted]

$T(X_{n+1}) \to T(X_n)$ and $T(X_n) \to T(X_{n-1})$

[redacted]

$T(X_{n+1}) \to T(A_{n+1}) \to T(X_n)$ and $T(X_n) \to T(A_n) \to T(X_{n-1})$

[redacted]

$$T(X_{n+1}) \to T(X_n) \to T(X_{n-1})$$

[redacted]

— ~+Xt^X1^X0^Z^ 0 (10.15.2)

and
0-»-.Z->-.X_1->-X_2->-•••,(10.15.3)

where ..., X0, X_v X_2, ... are all G-free and the mappings are G-homomorphisms. If, in this situation, we define X0->X_1 as the combined mapping X0-^Z-> X_v then the sequence

> X2 -*■ Xt -> X0 -» X_t X_2 -> • • • (10.15.4)

is exact. In view of this, it is convenient to represent the complete resolution by the single commutative diagram

• • X^~-*■ XQ—*'X_^-*'X_2~*" • •
V (10.15.5)
0 o

Suppose now that (10.15.5) is a complete G-free resolution of Z and let A be a left G-module. Denote by X the complex (10.15.4), then the homology module .H"{HomG(X, A)} is a covariant functor of A on account of the fact that each G-homomorphism A->A' produces a translation HomG (X, A) ->HomG (X, A'). Furthermore, if

0-+A*^A^A'->0

is an exact sequence of G-modules then, by Theorem 3 of section (5.1),

0 -> HomG (X, A *) HomG (X, A)-> HomG (X, A') 0

is an exact sequence of complexes. This, in turn, gives rise to the exact sequence

• • • ^ Hn{ HomG (X, A*)} -> Hn{ HomG (X,4)}-> Hn{ HomG (X, A')}
->i/n+1{Hom0 (X,yl*)}->Jff"+1{Hom&, (X, A)}->••.

$$\cdots \to X_2 \to X_1 \to X_0 \to Z \to 0 \qquad (10.15.2)$$

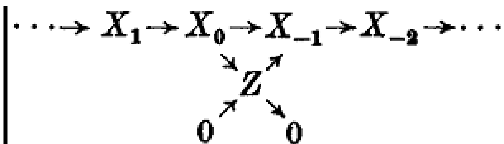
[redacted]

$$0 \to Z \to X_{-1} \to X_{-2} \to \cdots, \qquad (10.15.3)$$

[redacted]

$$\cdots \to X_2 \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots \qquad (10.15.4)$$

[redacted]



(10.15.5)

[redacted]

$$0 \to A^* \to A \to A' \to 0$$

[redacted]

$$0 \to \mathrm{Hom}_G(\mathbf{X}, A^*) \to \mathrm{Hom}_G(\mathbf{X}, A) \to \mathrm{Hom}_G(\mathbf{X}, A') \to 0$$

[redacted]

$$\cdots \to H^n\{\mathrm{Hom}_G(\mathbf{X}, A^*)\} \to H^n\{\mathrm{Hom}_G(\mathbf{X}, A)\} \to H^n\{\mathrm{Hom}_G(\mathbf{X}, A')\}$$
$$\to H^{n+1}\{\mathrm{Hom}_G(\mathbf{X}, A^*)\} \to H^{n+1}\{\mathrm{Hom}_G(\mathbf{X}, A)\} \to \cdots$$

of homology modules. Indeed, we can sum up these remarks and extend them by saying that ..., if_1{HomG(X, A)}, tf0{Homo(X,^)}, F{HomG(X,^)}, ... (10.15.6)	[redacted] $\ldots, H^{-1}\{\mathrm{Hom}_G(\mathbf{X}, A)\}, H^0\{\mathrm{Hom}_G(\mathbf{X}, A)\}, H^1\{\mathrm{Hom}_G(\mathbf{X}, A)\}, \ldots$ (10.15.6)
is an exact connected sequence of additive covariant functors. Theorem 20. Let G be a finite group and (10.15.5) a complete G-free resolution ofZ. Then the exact connected sequence (10.15.6) is isomorphic to the complete derived sequence of G.	[redacted]
Proof. Since (10.15.2) is a G-free resolution of Z, ^1{HomG(X,^4)} = Extb(Z,A) = J^G.A),	[redacted] $H^1\{\mathrm{Hom}_G(\mathbf{X}, A)\} = \mathrm{Ext}^1_G(Z, A) = J^1(G, A),$
hence (Proposition 14) it is enough to show that #n{HomG(X, ^4)} = 0 (—co<n<oo), (10.15.7)	[redacted] $H^n\{\mathrm{Hom}_G(\mathbf{X}, A)\} = 0 \quad (-\infty < n < \infty),$ (10.15.7)
whenever A is either G-projective or G-injective. By Theorem 17 this will be more than covered if (10.15.7) is established whenever A is G-special.	[redacted]
Assume therefore that A is G-special then (Proposition 12) it is isomorphic to a direct summand of A' = Homz (Z(G),A), where A’ has the structure of a left G-module obtained by regarding Z(G) as a right G-module. Now for any left G-module G we have isomorphisms-)- HomG(G, A') = HomG{G, TAomz{Z{G), A)} « Horn z{Z(G) ®aC,A} as Homz(G,^4),	[redacted] $\mathrm{Hom}_G(C, A') = \mathrm{Hom}_G\{C, \mathrm{Hom}_Z(Z(G), A)\} \approx \mathrm{Hom}_Z\{Z(G) \otimes_G C, A\}$ $\approx \mathrm{Hom}_Z(C, A),$
and this gives a functor equivalence between HomG((7,A') and Homz((7, A). It follows that corresponding homology modules of the two	[redacted] $\cdots \to \mathrm{Hom}_G(X_{n-1}, A') \to \mathrm{Hom}_G(X_n, A') \to \mathrm{Hom}_G(X_{n+1}, A') \to \cdots$

complexes
•   •• ->HomG (Xn_,, A') ->HomG (Xn, A') -^HomG (Xn+1, A') ->■•••
and
• • • ->Homz (Xn_v A) ->Homz (Xn,A)->Hom2 (Xn+1, A)-> — (10.15.8)
are isomorphic. But (10.15.4), being an exact sequence of G-free modules, is also an exact sequence of ^-modules hence, by Pro-position 15, (10.15.8) is exact. Accordingly //“{HomG(X, A')} = 0 and therefore Hn{HomG (X, A)} = 0 for all values of n.
Before we go on to establish the existence of complete free resolu¬tions of Z in the case of an arbitrary finite group, we shall illustrate the last theorem by considering the complete derived sequence of a finite cyclic group.
Let G be a cyclic group of order q and let <r be a generator. In this case Z(G) is a commutative ring whose general element has the form
9-1
2 nv<f,
K = 0
where, of course, the nv are integers. Put
N = 1 + a + ... + a-9-1 and T = cr — 1 (10.15.9) and consider the mappings
N   T
Z(G)->Z(G) and Z(G)^Z(G),
t See (8.5.4).
18   »HA
where the former consists of multiplication by N and the latter of multiplication by T.

If N('Ln,,<T’') = 0 then hnv = 0,

[redacted]

$$\cdots \to \mathrm{Hom}_Z(X_{n-1}, A) \to \mathrm{Hom}_Z(X_n, A) \to \mathrm{Hom}_Z(X_{n+1}, A) \to \cdots \qquad (10.15.8)$$

[redacted]

[redacted]

[redacted]

$$\sum_{\nu=0}^{q-1} n_\nu \sigma^\nu,$$

[redacted]

$$N = 1 + \sigma + \ldots + \sigma^{q-1} \quad \text{and} \quad T = \sigma - 1 \qquad (10.15.9)$$

[redacted]

$$Z(G) \overset{N}{\to} Z(G) \quad \text{and} \quad Z(G) \overset{T}{\to} Z(G),$$

[redacted]

[redacted]

because Na" = N, hence 9—1 a-l
2 nvdv = 2 n,,(a" — 1) = TA »=o
v=0
for a suitable AeZ(G). On the other hand, NT = 0, consequently T N
Z(G) Z(G) Z(G) is exact.
Suppose now that T(H,n,,a") = 0, then
(w0 + ?i1er+... + ng_1aa~1) — (n0a + n1a2 + ... -t-w^cr9) = 0,
and thereforen0 = n1= ... = nq_x. Thus Xw1,cr ’ = NA', where A' e Z(G),
N T
and so it is seen that Z(G)-^-Z(G)->Z(G) is also exact.
Consider next the augmentation homomorphism Z(G)->Z as defined in section (10.3). If crv belongs to its kernel, then
n0 + n j + = 0,
which, as we saw above, implies that hnvav is of the form AT. But every element of this form certainly belongs to the kernel, consequently T Z(G)->-Z(G)->Z-^-0 is exact. Finally, the O-homomorphism TZ^-Z(G), in which 1 Z->N, makes 0->Z->Z(G) ->Z(G) exact and, N moreover, Z(G)-+Z(G) can be represented as Z(G) Z ->■ Z(G). Collecting all these facts together we obtain
Theorem 21. Let G be a cyclic group of order q then NT N T N
Z(G)-* Z{G)^Z(G)»Z(G)-+ Z(G) -* Z(G) •..
X X Z * \
0 0
is a complete G-free resolution of Z. Here N and T, when used to indicate mappings, signify multiplication by the elements N = 1 + cr +... + crq~1 and T = a — 1 respectively. Z(G)-^Z is the usual augmentation map¬ping and, in Z-*Z{G), lz maps into N.

$$\sum_{\nu=0}^{q-1} n_\nu \sigma^\nu = \sum_{\nu=0}^{q-1} n_\nu(\sigma^\nu - 1) = T\lambda$$

[redacted]

$$(n_0 + n_1\sigma + \ldots + n_{q-1}\sigma^{q-1}) - (n_0\sigma + n_1\sigma^2 + \ldots + n_{q-1}\sigma^q) = 0,$$

[redacted]

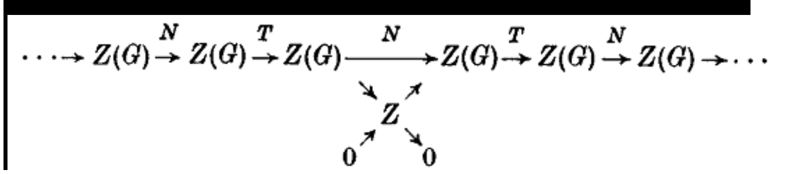
[redacted]

[redacted]

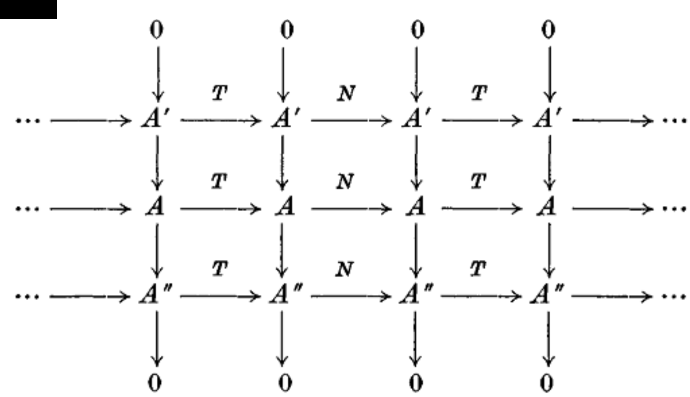
$$n_0 + n_1 + \ldots + n_{q-1} = 0,$$

[redacted]

[redacted]



[redacted]

Still supposing that G is cyclic, let A be a left G-module then, by Theorems 20 and 21, the complete derived sequence of G consists of the homology groups of a complex	[redacted] $\cdots \to \mathrm{Hom}_G(Z(G), A) \to \mathrm{Hom}_G(Z(G), A) \to \mathrm{Hom}_G(Z(G), A) \to \cdots.$
But Homfl (Z{G), A) « A and on identifying these two we obtain Theorem 22. Let G be a cyclic group of order q and A a left G-module. Then the complete derived sequence of G can be computed as the homology groups of the complex ……………………	[redacted] $\cdots \to A \xrightarrow{T} A \xrightarrow{N} A \xrightarrow{T} A \xrightarrow{N} A \to \cdots,$
where the mappings N consist of multiplication by 1 + cr +... + <r«_1 and the mappings T of multiplication by or —1. (T operates on the component modules with even indices.) Accordingly J2n(G, A) = AG/NA and J^+^G, A) = NAjIA.	[redacted] $J^{2n}(G, A) = A^G/NA \quad and \quad J^{2n+1}(G, A) = {}_NA/IA.$
Observe that an exact sequence 0->A' ^A^A"^0 of G-modules, gives rise to an exact sequence ■ ■■->NA"/IA" A'°jNA'-+A°/NA -> A"GjNA" ^NA'IIA'^NAjIA^NA"IIA"^A'GINA'->--	[redacted] $\cdots \to {}_NA''/IA'' \to A'^G/NA' \to A^G/NA \to A''^G/NA''$ $\to {}_NA'/IA' \to {}_NA/IA \to {}_NA''/IA'' \to A'^G/NA' \to \cdots$
and here the connecting homomorphisms are those obtained from the exact sequence of complexes.	[redacted]  [redacted]
Let us return to the consideration of a	[redacted]

general finite group. It will be convenient to prove two lemmas.

Lemma 2. Let M be a Z-free module with the elements//1,//2, ...,/isas a base and let fa : M -> Z (1 < i ^ s) be the Z-homomorphism defined by

(i=j),

(i+j)-

Then <f>v 02, ...,<j>s are a Z-basefor Homz (M, Z). The verification is immediate.

[redacted]

[redacted]

$$\phi_i(\mu_j) = \begin{cases} 1_Z & (i = j), \\ 0 & (i \neq j). \end{cases}$$

[redacted]