Spanning cubic graph designs	Các thiết kế đồ thị bậc ba sinh (đồ thị bậc ba khung) Bài báo cáo mà anh nói nằm trong đây: trang 367, mã số báo cáo VII-O-0.1 http://www2.hcmuaf.edu.vn/data/quoctuan/TapTomTatBaoCao-Thuoc-Final-101108.pdf
Graph designs are natural extensions of BIBDs (balanced incomplete block designs). In this paper we explore spanning cubic graph designs and develop tools for constructing some of them. We show that K16 can be decomposed into each of the 4060 connected cubic graphs of order 16, and into precisely 144 of the 147 disconnected cubic graphs of order 16. We also identify some infinite families of cubic graphs of order 6n + 4 that decompose K6n+4.	Các thiết kế đồ thị là sự mở rộng tự nhiên của các BIBD (thiết kế khối không đầy đủ được cân bằng). Trong bài báo này, chúng tôi nghiên cứu các thiết kế đồ thị bậc ba sinh (đồ thị bậc ba khung) và phát triển các công cụ để xây dựng một trong số chúng. Chúng tôi chứng tỏ rằng K16 có thể được phân rã thành một trong 4060 đồ thị bậc ba liên thông order 16, và thành 144 trong số 147 đồ thị bậc ba không liên thông order 16. Chúng tôi cũng xác định vô số họ đồ thị bậc ba order 6n + 4 phân rã K6n 4. Nếu dịch Order là bậc thì "cubic graphs of order 16" sẽ thành "đồ thị bậc ba bậc 16" có vẻ không ổn.
1. Introduction We say that a graph G decomposes the complete graph Kn if the edges of Kn can be covered by edge-disjoint copies of G. Such a covering is then called a decomposition of Kn into (copies) of G. This notion was first introduced by Hell and Rosa [9], and is a natural extension of BIBDs	[redacted] [redacted]

(balanced incomplete block designs) in which blocks (complete subgraphs) are replaced by another graph G. Following BIBD notation, we use the triple (n, G, 1) to indicate that the graph G decomposes Kn .If G has n vertices we call (n, G, 1) a spanning graph design.	[redacted]
In this note we shall limit our discussion to spanning decompositions of Kn. The oldest spanning decomposition is probably Kirkman's [14] proof that all 1-regular graphs of order 2n decompose K2n. This topic is still popular today; see for example the survey paper [17] and the book [25].	[redacted] [redacted]
The well-known Oberwolfach problem deals with decomposing Kn into a spanning 2-regular graph. It has only been solved for sporadic families of graphs [4,6,10,11]. When the 2-regular graph is required to be connected, i.e., Hamilton cycles, then the obvious arithmetic conditions, that is n divides the number of edges of Kn, and n is odd, are also sufficient. This follows from Walecki's famous decomposition of K2k+1 into k Hamilton cycles, as described by Lucas [15].	[redacted]
Decompositions of Kn into spanning	[redacted]

cubic (3-regular) graphs have been considered in [3,24]. Interestingly, it is well known that the Petersen graph does not decompose K10 [8,23]. Consequently, most of the research concentrated on decompositions of small complete graphs into cubic graphs. Imrich [12] proved that there are only 21 distinct cubic graphs of order 10. Adams, Bryant and Khodkar [2] proved that fifteen of the 21 graphs decompose K10 while the other six do not. Khosrovshahi et al. [13] extended this work by an extensive computer search and produced a table of all possible

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Fig. 1. A class of planar Hamiltonian cubic graphs G6n+4.

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Fig. 2. The six cubic graphs of order 10 that do not decompose K10.

decompositions of K10 into three (not necessarily isomorphic) cubic graphs. For each triple of cubic graphs G1, G2, G3 they also included a count of how many non-isomorphic decompositions of K10 into G1, G2, G3 exist. Similar results for K10 were also obtained by Petrenjuk [18,19].

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Ringel's conjecture that every tree of order $n + 1$ decomposes K_{2n+1} [20] is probably the most famous open graph design problem. Kotzig called it the Graph Disease. Various labelings are powerful tools to tackle such decompositions. They were introduced by Rosa in [21]. His ft-valuations were later renamed graceful labelings.

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In this note we concentrate on cubic decompositions of complete graphs and try to extend the results mentioned above to cubic graphs of arbitrary size. Clearly, if a cubic graph G forms a spanning graph design (k, G, 1) then $k = 6n + 4$ for some $n > 1$. (The case $n = 0$ is trivial.) So our initial question was whether for each $n > 1$ there are cubic graphs G of order $6n + 4$ that decompose K_{6n+4}. One obvious approach was to use cyclic 1-factorizations. For each positive integer n this quickly led to cubic graphs of order $6n + 4$ that decompose K_{6n+4}. Then we decided to look for cubic graphs that do not decompose K_{6n+4}. Using computer search we found that all but three cubic graphs of order 16, and all those we considered of orders 22 and 28, decomposed the corresponding complete graph. This suggests that spanning cubic graphs that do not

[redacted]

decompose K6n+4 are rare.

2. Spanning cubic graph designs

In this section we develop tools to construct spanning cubic graph designs.

As noted in [3], using well-known 1-factorizations of K2m, m = 3n+2, it is easy to construct examples of cubic graphs that decompose K6n+4.

For instance, the planar, Hamiltonian cubic graph in Fig. 1 is obtained from the well-known 1-factorization GK2m (see [17]) defined by:

- M0 = {(to, 0), (1, 2m — 2),..., (i, 2m — i — 1),..., (m — 1, m)}

- Mk = {(i + k, j + k) | (i, j) e M0}, k = 1,..., 2m — 2, with all arithmetic done mod (2m — 1), and to + k = to.

In what follows, we use a labeling scheme that turns out to be very powerful in finding many graphs that decompose K6n+4. We discuss two approaches: Breadth first search (BFS) and Depth first search (DFS). In BFS we search all cubic graphs of a fixed order. In DFS we search

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infinitely long sequences of cubic graphs of the same ‘‘type’’.	[redacted]
2.1. The cubic labeling, BFS	[redacted]
Our first attempt was to search for some ‘‘obvious’’ cubic graphs of order 16 that fail to decompose K16.To do so, we can start with the six cubic graphs of order 10 that fail to decompose K10, c.f [2], (see Fig. 2): Two of the graphs (G2 and G3) are bipartite and their union has chromatic number <8. The graph G1 is the Petersen graph, which fails to decompose K10 for many reasons. The graph G6 contains K4 but its independence number is 3. The argument that the remaining two graphs fail to decompose K10 relies on the fact that we are trying to decompose a complete graph into three subgraphs.	[redacted]
	[redacted]
	[redacted]
First we tried to extend the six cubic graphs that failed to decompose K10 to cubic graphs of order 16. The first few graphs we checked included natural generalizations of these graphs and also a cubic graph of chromatic index 4; see Fig. 3(a), Figs. 4 and 5.	[redacted]

	[redacted]
For each graph G, a decomposition of K16 was found with a similar structure. Let the vertices of K16 be labeled by {to} U A0 U A1 U A2 where Ai = {0i,..., 4i}. In each case, a cyclic starter graph G0 was found, with G0 = G. The other four disjoint isomorphic copies Gj were obtained by the simple mapping <fij(xk) = (x + j mod 5)k for j e {1, 2, 3, 4} and k e {0, 1, 2}, and jTO) = to. The graphs Gj, 0 < j < 4, together decomposed K16 because the labels on G0 were chosen such that:	[redacted]
• For each 0 < i < 2, to was connected to exactly one vertex in Ai, and there were exactly two edges between vertices spanned by each Ai.	[redacted]
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Fig. 3. Index 4 cubic graph G1 with a cubic labeling (left), and its standard form cubic labeling (right).	[redacted]
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Fig. 4. The graph DG16 and its computer generated labeling.	[redacted]
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Fig. 5. Generalized Petersengraphs.	[redacted]

• Exactly one of these edges has length in {1, 4} (mod 5), and the length of the other edge is in {2, 3} (mod 5). • Between Ai and Aj, j > i there were exactly 5 edges (xi, yj) such that {(y — x)} (mod 5) = {0,1, 2, 3, 4}. • G0 is a cubic graph. This implied that not only did each of these graphs decompose K16 cyclically, but in each case there was a decomposition in which one vertex remained fixed. A computational search verified that all decompositions of the fifteen cubic graphs of order 10 also had a fixed point. This led to the following question: Is it true that all cubic graphs of order 6n + 4, n > 2, decompose K6n+4 cyclically? It turned out that the answer is negative, even though it is almost always true for n = 2: only twelve of the 4207 cubic graphs of order 16 do not decompose K16 cyclically. All of these are disconnected graphs. Hence we ask: Is it true that all connected cubic graphs of order 6n + 4, n > 2, decompose K6n+4 cyclically? Among the graphs tested, and for which there exists a cyclic	[redacted]

decomposition of K16, are G1, shown in Fig. 3, the generalized Petersen graphs shown in Fig. 5, K4 + 2PS3 (the symbol + represents graph union) and PS3 is the 3-prism, and DG16 shown in …………………………………………………	[redacted]
Fig. 6. The graph G8.	[redacted]
Fig. 4. A closer study of these decompositions revealed that within each Ai we may shift the labels of the vertices by a fixed amount without affecting the decomposition. This led us to the following standard form cubic labeling:	[redacted]
(i) LetAi = {0i,..., (2n)i}, i = 0, 1, 2.	[redacted]
(ii) Let V(G0) = {to} U A0 U A1 U A2.	[redacted]
(iii) Let E(G0) = {(to, 00), (to, 01), (to, 02)} U E0 U E1 U E2 U £0,1 U £1,2 U £0,2.	[redacted]
(iv) Ei is a set of n edges (xi, yi) such that all differences {±(x — y) mod (2n + 1)} = {1, 2,... 2n}.	[redacted]
(v) Ei,j is a set of 2n + 1 edges {(xi, y)} such that {(x — y) (mod 2n + 1)} = {0, 1,..., 2n}.	[redacted]
(vi) G0 is a cubic graph.	[redacted]
Fig. 3(a) shows the computer	[redacted]

generated labeling of a graph, and Fig. 3(b) shows the same graph relabeled in standard form.	[redacted]
Given the labeled graph G0, define the graphs Gi, i = 1,..., 2n as follows:	[redacted]
• V (Gi) = V (G0).	[redacted]
• (xk, yj) e E(Gi) iff ((x — i)k, (y — i)j) e G0, where arithmetic is mod 2n + 1.	[redacted]
It is easy to see that the mappings <i(xk) = (x + i mod (2n + 1))k and <^(to) = to, for i = 1,..., 2n, map G0 onto 2n isomorphic, pairwise edge-disjoint, cubic graphs on the same set of vertices, yielding the spanning graph design (6n + 4, G0, 1). Following design theory custom, we call these decompositions cyclic decompositions.	[redacted]
It is clear that the freedom of choosing the edges within each Ei and Ei,j promises a very large number of cubic graphs of order 6n + 4 that decompose K6n+4. The cubic labeling in Fig. 4 can be readily implemented in programs that generate spanning cubic graph decompositions. At this stage we embarked on a BFS of all cubic graphs of order 16. We independently wrote two programs that complemented each other. In the	[redacted]

first, cubic graphs were picked from Brendan McKay's list [16] and the program tried to fit them with the cubic labeling. In the other program we started by generating all possible standard form cubic labelings and then matching these graphs with those in Brendan's list. The results were somewhat surprising. Almost all cubic graphs of order 16 cyclically decomposed K16. Specifically, all 4060 connected cubic graphs of order 16 decompose K16 cyclically. Of the 147 disconnected cubic graphs of order 16,135 decompose K16 cyclically, nine decompose K16 non-cyclically, and three fail to decompose K16. The website [1] contains a list of all cubic graphs of order 16, with standard form cubic labelings when they exist, other decomposition descriptions and the three failed graphs. The listing of the graphs is in the same order as in McKay's list [16]. We must add that our claim that three graphs do not decompose K16 (namely K4 + K3 3 + PS3,2K4 + G8 (cf. Fig. 6) and K3 3+ the Petersen graph) are computational proofs. We could not come up with a mathematical argument to substantiate this claim. 2.2. The cubic labeling, DFS	[redacted]

In this section we use a number of approaches to demonstrate the power of the standard cubic labeling to generate infinite sequences of spanning cubic graph designs. The first such infinite sequence was constructed by Hanani et al. [7]. They constructed resolvable BIBDs B[4,1; v] for all v = 4 mod 12, thus proving that the cubic graph consisting of 3k + 1K4's decomposes K12k+4. A second family was constructed by Adams et al. [3] where cubic graphs consisting of disjoint copies of the 3-cube were shown to decompose K16+24k.	[redacted]
For our first example let Tn be the sequence of recursively defined planar cubic graphs:	[redacted]
• Let T0 = K4 be embedded in the plane with one vertex, say 0, inside the outer triangle.	[redacted]
• Assume that Tn—1 is embedded in the plane so that its outer face is a triangle.	[redacted]
• To construct Tn subdivide each of the three edges of the outer face of Tn—1 by a vertex. For each of these three vertices add a new vertex (into the outer face of Tn—1) and join it to the corresponding	[redacted]

vertex. Finally add three edges joining the vertices of degree one. The three added vertices can be embedded into the outer face of T_n—1 so that the resulting graph is planar.	[redacted]
Clearly T_n has $6n + 4$ vertices, it has one triangular face (the outer face), three 4-faces (around the center 0), three 5-faces, and the remaining faces are hexagons; see Fig. 7.	[redacted]
………………………………………………………………	[redacted] [redacted]
	[redacted]
Fig.7. Thegraph T_n.	[redacted]
Theorem 2.1. For every $n > 1$, T_n decomposes K_{6n+4}.	[redacted]
Proof. Let $1 < x < n$ and let x_i e A_i (A_i is as defined in the standard form cubic labeling.) The following cubic labeling generates this graph:	[redacted]
• In each E_i select n edges: (x_j, —x_i).	[redacted]
• In $E_{i,i+1}$ select the perfect matching $\{(x_j, (-x + 1)_{j+1})\}$ (with all arithmetic done modulo $2n + 1$ on the vertex labels and mod 3 on the indices).	[redacted]
Clearly, the resulting graph is a cubic graph. Since — $(n + 1) + 1$ = — n = $n + 1$ (mod $2n + 1$) the vertices {(n +	[redacted]

1)0, (n + 1)1, (n + 1)2} form a triangle, the outer triangle in Fig. 7. The following alternative description of the graph Tn will help. Tn consists of n internal cocentric hexagons {C1,..., Cn}. C1 = {00, 11, 02, 10, 01, 12} and the ‘‘inward’’ edges (0k, to), k = 0, 1, 2. The hexagons Cj = {i0, (2n + 2 — i)1, i2, (2n + 2 — i)0, i1, (2n + 2 — i)2}, i = 2,..., n have ‘‘inward’’ edges (ik, (2n + 1 — i)k), k = 0, 1, 2, connecting them to the hexagons Cj— 1. The vertices of the outer triangle {(n + 1)0, (n + 1)1, (n + 1)2} are connected by an edge to the vertices {n0, n1, n2} of the hexagon Cn. All these edges are the edges of the cubic labeling. Fig. 7 provides a visual proof of the theorem. □

Another infinite sequence of spanning cubic graph designs is the sequence K4 + nPS3 (where PS3 is the 3-prism). It is known that K4 + PS3 does not decompose K10. We have:

Theorem 2.2. For every n ^ 1 (mod 3), K4 + nPS3 decomposes K6n+4.

Proof. We give a short description of the applicable cubic labeling and leave the details to the reader. Let 1 < x < n and let xi e Ai (Ai is as defined in the standard form cubic labeling.) We start with the edges

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{(xi, —xi)} in each Ei. We add the edges E01 = {(x0, (2x)1)}, E12 = {(x1, (nx)2)} and E2,0 = {(x2, —x0)}.The set {to, 00, 01, 02} spans a K4. The vertices

[redacted]

{x0, (—x)0, (2x) 1, (—2x)1, (2nx)2, (—2nx)2}

[redacted]

span a copy of PS3 (note that 2nx = —x mod (2n + 1)). It is easy to check that this is a proper labeling and G0 = K4 + nPS3. If n = 3k + 1 then gcd (2n + 1, 2n — 2) = 3 and the edges in E12 will not form a cubic labeling, (all differences will be multiples of 3). □

[redacted]

In Fig. 8 we show a standard form cubic labeling of K4 + 4PS3. We also found a standard form cubic labeling for K4 + 7PS3. These results led us to:

[redacted]

Conjecture 2.1. For every n > 1, K4 + nPS3 decomposes K6n+4.

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Here is another example of a simply stated DFS problem. Let DG2n be the cubic graph C2n plus the main diagonals, see Fig. 4. As noted in [2,13], DG10 does not decompose K10. On the other hand, DG16 and DG22 cyclically decompose K16 and K22 respectively. They admit a cubic

[redacted]

labeling as in Fig. 9.	[redacted]
We could not find a way to generalize these labelings but we conjecture:	[redacted]
Conjecture 2.2. For all n > 1 , DG6n+4 decomposes K6	[redacted]
A similar graph, K4 +DG6n could be handled with a different kind of labeling, using group elements. The following theorem demonstrates this.	[redacted]
Theorem 2.3. If 2n + 1 = pr (p prime) and gcd(n, 3) = 1 then K4 + DG6n decomposes K6n+4.	[redacted]
Proof. Let 1 < x < n and let xi e Ai (Ai is as defined in the standard form cubic labeling.) Let a be a primitive root in GF(pr). Label the vertices and edges of a graph G of order 6n + 4 as follows:	[redacted]
• The vertices {TO, 00, 01, 02} form a copy of K4.	[redacted]
• Ai = {1i, ai, a2, ..., a2n— 1}, i = 0, 1, 2.	[redacted]
• In each Ai add a matching consisting of the edges (xi, (-x)i).	[redacted]
• Add the edges {(xi, (ax)i+1)}, where the index is taken modulo 3.	[redacted]

Clearly, G is a cubic graph. Since ax — x = ay — y and a2x — x = a2y — y, if x = y this labeling is a standard form cubic labeling of the cubic graph G and thus it decomposes K6n+4. So we need to show that G = K4 + DG6n. As noted above, {to, 00, 01, 02} forms a copy of K4.

Consider the sequence:
S = {10, a1, a22, a0, a4, ajj, ..., a2n, ..., a^1}.

We first note that the vertices in A0 appear in this sequence in every third position, that is, as a03k. Similarly, A1 appears in the subsequence af^1 and A2 in the subsequence a3<+2. Also since gcd(n, 3) = 1, a2n = 1, a4n = 1 will appear in the sequence with subscripts 0, 1, 2. Similarly, it can be easily seen that the sequence contains all vertices in A0 U A1 U A2. Also ak is connected by an edge to a-ij+11. Finally, a6n—1 = a—1 and a—1 is connected by an edge to 10. Thus the sequence S forms a cycle of length 6n. Since a is a primitive root modulo 2n + 1, an = —1. For every vertex a*k, the vertex aln++ = — aJk is at distance 3n from it on the cycle S. But these 2 vertices are connected by an edge and hence S spans a subgraph isomorphic to DG6n. □

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Conjecture 2.3. For every n > 1, DG6n + K4 decomposes K6n+4.	[redacted]
We were able to identify other sequences of spanning cubic graph designs. These theorems demonstrated three different samples: a complete sequence, a partial but infinite sequence and using groups for labeling. Other sequences, like the cubic graphs DG6n+4, are still waiting on the decomposition pile.	[redacted]
2.3. Concluding remarks	[redacted]
By [5], the following decision problem is in NP:	[redacted]
Input: A cubic graph G of order 6n + 4.	[redacted]
Output: TRUE if G decomposes K6n+4.	[redacted]
However, is the following problem also in NP?	[redacted]
Input: A cubic graph G of order 6n + 4.	[redacted]
Output: TRUE if G does not decomposes K6n+4.	[redacted]
We were not able to find ‘‘constructive’’ proofs or find ideas for proving that cubic graphs fail to decompose K6n+4, even for a single graph of order 16.	[redacted]
A closely related problem is how	[redacted]

many edge-disjoint copies of a given cubic graph G of order 2n can fit inside K2n. It follows from a theorem of Sauer and Spencer [22] that if G has at least 18 vertices then at least two edge-disjoint copies of G can fit inside K2n. We conjecture:	[redacted]
Conjecture 2.4. If G is a cubic graph of order 2n, n > 7, then G can cover at least 2/3 of the edges of K2n.	Phỏng đoán (giả định) 2.4. Nếu G là một đồ thị bậc ba order 2n, n> 7, thì G có thể bao phủ ít nhất 2/3 số cạnh của K2n.